Free vibration analysis of generalized thermoelastic doubly connected plate of polygonal shape

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Abstract. Free vibration analysis of generalized thermoelastic doubly connected polygonal plate is studied using the Fourier expansion collocation method. The equation of motion based on two-dimensional theory of elasticity is applied under the plane strain assumption of generalized thermoelastic plate of polygonal shape composed of homogeneous isotropic material. The frequency equations are obtained by satisfying the boundary conditions along the inner and outer surface of the polygonal plate. The numerical calculations are carried out for triangular, square, pentagonal and hexagonal plates. The computed non-dimensional frequencies are compared with the Lord-Shulman (LS), Green-Lindsay (GL), coupled theory (CT) and uncoupled theory (UCT) theories of thermoelasticity and they are presented in tables. The dispersion curves are drawn for longitudinal and flexural antisymmetric modes of vibration.

Keywords: Wave propagation in plate, vibration of thermal plate, piezoelectric plate, plate immersed in fluid, generalized thermoelastic plate, thermal stress, mechanical vibrations.

INTRODUCTION

The wave propagation in plates of circular and plates of various shapes are often used as structural components and their vibration characteristics are important for practical design. The propagation of waves in thermoelastic materials has many applications in various fields of science and technology, namely, atomic physics, industrial engineering, thermal power plants, submarine structures, pressure vessels, aerospace, chemical pipes, and metallurgy. The importance of thermal stresses in causing structural damages and changes in functioning of the structure is well recognized whenever thermal stress environments are involved. Therefore, the ability to predict elastodynamics stresses induced by sudden thermal loading in composite structures is essential for the proper and safe design, and the knowledge of its response during the service in these severe thermal environments.

A method for solving wave propagation in polygonal plates and to find out the phase velocities in different modes of vibrations, namely, longitudinal, torsional and flexural, by constructing frequency equations was devised by (Nagaya, 1981a, b, c, 1983a, b). He formulated the Fourier expansion collocation method for this purpose and the same method is used in this paper. The generalized theory of thermo elasticity was developed by Lord and Shulman (1967) involving one relaxation time for isotropic homogeneous media, which is called the first generalization to the coupled theory of elasticity. These equations determine the finite speeds of propagation of heat and displacement distributions. The corresponding equations for anisotropic case were obtained by Dhalwal and Sherief (1980).

The second generalization to the coupled theory of thermoelasticity is what is known as the theory of thermoelasticity, with two relaxation times or the theory of temperature-dependent thermoelasticity. A generalization was proposed by Green and Laws (1972). Green and Lindsay (1972) obtained an explicit version of the


In this paper, free vibration of generalized thermoelastic doubly connected polygonal plate composed of homogeneous isotropic material is studied using the Fourier expansion collocation method based on Lord-Shulman (LS), Green-Lindsay (GL), coupled theory (CT) and uncoupled theory (UCT) theories of thermoelasticity. The solutions to the equations of motion for an isotropic medium is obtained by using the two dimensional theory of elasticity. It is assumed that, there is no vibration and displacement along the z-axis, that is the displacement along the z-axis, \( w \) is zero. The computed non-dimensional frequencies are compared with Lord-Shulman (LS), Green-Lindsay (GL), coupled theory (CT) and uncoupled theory (UCT) theories of thermoelasticity for longitudinal and flexural antisymmetric modes of vibrations and the frequencies are tabulated. The dispersion curves are drawn for longitudinal and flexural antisymmetric modes of vibrations.

**FORMULATION OF THE PROBLEM**

We consider a homogeneous, isotropic, thermally conducting elastic doubly connected plate of polygonal plate with uniform temperature \( T_0 \) in the undisturbed state initially. The system displacements and stresses are defined by the polar coordinates \( r \) and \( \theta \) in an arbitrary point inside the plate and denote the displacements \( u_r \) in the direction of \( r \) and \( u_\theta \) in the tangential direction \( \theta \). The in-plane vibration and displacements of doubly connected polygonal plate are obtained by assuming that there is no vibration and displacement along the z-direction in the cylindrical coordinate system \((r, \theta, z) \). The two dimensional stress equations of motion, strain displacement relations and heat conduction equation in the absence of body forces for a linearly elastic medium are considered from Sharma and Sharma (2002) as:

\[
\begin{align*}
\sigma_{rr} &+ r^{-1} \sigma_{r\theta} + r^{-1} \left( \sigma_{\theta\theta} - \sigma_{rr} \right) = \rho u_r, \\
\sigma_{r\theta} &+ r^{-1} \sigma_{\theta\theta} + 2r^{-1} \sigma_{\theta\theta} = \rho u_\theta \\
K \left( T_r + r T_r' + r^2 T_{r\theta} \right) - \rho c_r \left( \dot{T}_r + r \dot{T}_r' \right) &+ \rho \left( \ddot{u}_r + r \ddot{u}_r + \ddot{u}_\theta \right) = 0
\end{align*}
\]

and

\[
\begin{align*}
\sigma_{rr} &= \lambda \left( e_{rr} + e_{\theta\theta} \right) + 2\mu e_{rr} - \beta \left( T + t \delta_{2z} \dot{T} \right) \\
\sigma_{r\theta} &= \lambda \left( e_{rr} + e_{\theta\theta} \right) + 2\mu e_{r\theta} - \beta \left( T + t \delta_{2z} \dot{T} \right) \\
\sigma_{\theta\theta} &= 2\mu e_{\theta\theta}
\end{align*}
\]

where \( \sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \) are the stress components, \( e_{rr}, e_{\theta\theta}, e_{r\theta} \) are the strain components, \( \rho \) is the mass density, \( \lambda, \mu \) are the Lamé parameters, \( \beta \) is the thermal expansion coefficient, \( K \) is the thermal conductivity, and \( T \) is the temperature. The coupled thermoelastic effects are considered through the coupling parameters \( \lambda, \mu, \beta \) and the thermal expansion coefficient \( \beta \). The system displacements and stresses are defined by the polar coordinates \( r \) and \( \theta \) in an arbitrary point inside the plate and denote the displacements \( u_r \) in the direction of \( r \) and \( u_\theta \) in the tangential direction \( \theta \).
are the strain components, \( T \) is the temperature change about the equilibrium temperature \( T_0 \), \( \rho \) is the mass density, \( c_i \) is the specific heat capacity, \( \beta \) is the thermal capacity factor that couples the heat conduction and elastic field equations, \( K \) is the thermal conductivity, \( t_0, t_i \) are the two thermal relaxation times, \( t \) is the time, \( \lambda \) and \( \mu \) are Lamé’s constants. The comma notation is used for spatial derivatives; the superposed dot represents time differentiation, and \( \delta_{ij} \) is the Kronecker delta. In addition, \( k = 1 \) for Lord-Shulman (LS) theory and \( k = 2 \) for Green-Lindsay (GL) theory. The thermal relaxation times \( t_0 \) and \( t_i \) satisfy the inequalities \( t_0 \geq t_i \geq 0 \) for GL theory only and we assume that \( \rho > 0, T_0 > 0 \), and \( c_i > 0 \). The strain \( e_{ij} \) related to the displacements is given by:

\[
e_{rr} = u_{r,r}, \quad e_{\theta \theta} = r^{-1}(u_r + u_{\theta,\theta}), \quad e_{\phi \phi} = u_{\phi,\phi} - r^{-1}(u_\phi - u_{\phi,\theta})
\]

(5)

**LORD-SHULMAN (LS) THEORY**

The Lord-Shulman theory of heat conduction equation for a two-dimensional theory of thermoelasticity is obtained by substituting \( k = 1 \) in Equations 2 and 3, and are considered from Sharma and Sharma (2002) as:

\[
K(T_{rr} + rT_r + rT_\theta) - \rho c_r (T + t_i T) = T_{tt} \left( \frac{\partial}{\partial r} + t_0 \frac{\partial}{\partial r^2} \right) \left[ \beta \left( u_{rr} + r^{-1}(u_{\theta \theta} + u_r) \right) \right]
\]

(6)

and

\[
\sigma_{rr} = \lambda (e_{rr} + e_{\theta \theta}) + 2\mu e_{rr} - \beta T
\]

\[
\sigma_{\theta \theta} = \lambda (e_{rr} + e_{\theta \theta}) + 2\mu e_{\theta \theta} - \beta T
\]

(7)

Substituting Equations 4, 5 and 7 in Equations 1 and 6, the displacement equations of motions are obtained as:

\[
(\lambda + 2\mu)\left( u_{rr} + r^2 u_r - r^3 u_r \right) + \mu r^3 u_{rr} + r^2 (\lambda + \mu) u_{\phi \phi} + r^2 (\lambda + 3\mu) u_{\theta \theta} - \beta T = \rho \ddot{u}_r
\]

\[
\mu\left( u_{rr} + r^2 u_r - r^3 u_r \right) + r^2 (\lambda + 2\mu) u_{\phi \phi} + r^2 (\lambda + 3\mu) u_{\theta \theta} - \beta T = \rho \ddot{u}_\theta
\]

\[
K(T_{rr} + rT_r + rT_\theta) - \rho c_r (T + t_i T) = T_{tt} \left( \frac{\partial}{\partial r} + t_0 \frac{\partial}{\partial r^2} \right) \left[ \beta \left( u_{rr} + r^{-1}(u_{\theta \theta} + u_r) \right) \right]
\]

(8)

**Solution of the problem**

Equation 8 is a coupled partial differential equation with two displacements and heat conduction components. To uncouple Equation 8, we follow the solutions by Mirkys (1964) by assuming that the vibration and displacements along the axial direction \( z \) is equal to zero, and assuming the solutions of the Equation 8 in the form:

\[
u_r (r, \theta, t) = \sum_{n=0}^{\infty} e_n \left[ \left( \phi_n \alpha + r^{-1} \psi_n \beta \right) + \left( \tilde{\phi}_n \alpha + r^{-1} \tilde{\psi}_n \beta \right) \right] e^{i\omega t}
\]

\[
u_\theta (r, \theta, t) = \sum_{n=0}^{\infty} e_n \left[ \left( r^{-1} \phi_n \alpha - \psi_n \beta \right) + \left( r^{-1} \tilde{\phi}_n \alpha - \tilde{\psi}_n \beta \right) \right] e^{i\omega t}
\]

\[
T (r, \theta, t) = \left( \lambda + 2\mu / \beta a^2 \right) \sum_{n=0}^{\infty} \left( T_n + \tilde{T}_n \right) e^{i\omega t}
\]

(9)

Where \( e_n = \frac{1}{2} \) for \( n = 0 \), \( e_n = 1 \) for \( n \geq 1 \), \( i = \sqrt{-1} \), \( \omega \) is the angular frequency, \( \phi_n (r, \theta), \psi_n (r, \theta), \tilde{\phi}_n (r, \theta), \tilde{\psi}_n (r, \theta) \) and \( \tilde{T}_n (r, \theta) \) are the displacement potentials.

Introduce the dimensionless quantities such as:

\[
\bar{\lambda} = \lambda / \mu, \quad x = r / a, \quad \tau_0 = (1 + t_i i \omega), \quad \bar{K} = K \sqrt{\rho / t_0} \beta a T_0 \Omega, \quad \bar{d} = \rho c_r / \beta \mu a T_0, \quad T_n = \tau_0 / (1 + t_i i \omega),
\]

\( \bar{c}_0^2 = (\lambda + 2\mu) / \rho \), \( \Omega^2 = \omega^2 a^2 / c_0^2 \), \( c_0^2 \) is the phase velocity substituting Equation 9 in Equation 8, we obtain:

\[
\left\{ (2 + \bar{\lambda} \bar{K}^2 + \Omega^2) \phi_n - T_n = 0
\right\}
\]

\[
\left\{ \bar{V}^2 \phi_n + \left( i \bar{K} \bar{V}^2 + \bar{d} \right) T_n = 0
\right\}
\]

(10)

and

\[
\left\{ \bar{V}^2 + \Omega^2 \right\} \psi_n = 0
\]

(11)

where \( \bar{V}^2 = \partial^2 / \partial x^2 + x^{-1} \partial / \partial x + x^2 \partial^2 / \partial \theta^2 \)

Eliminating \( T_n \) from Equation 10, we obtain:

\[
\left( A \bar{V}^4 + B \bar{V}^2 + C \right) \phi_n = 0
\]

(12)

where

\[
A = i \bar{K} \left( 2 + \bar{\lambda} \right), \quad B = \left\{ (2 + \bar{\lambda}) \bar{d} + i \bar{K} \Omega^2 + 1 \right\}, \quad C = \Omega^2 \bar{d}.
\]

(13)

in which \( A, B \) and \( C \) are arbitrary constants and are used to find the roots of Equation 12.

The solution of Equation 12 for the symmetric mode is:

\[
\phi_n = \sum_{i=1}^{2} \left[ A_m J_n (\alpha, ax) + B_m Y_n (\alpha, ax) \right] \cos n \theta
\]

\[
T_n = \sum_{i=1}^{2} a_i \left[ A_m J_n (\alpha, ax) + B_m Y_n (\alpha, ax) \right] \cos n \theta
\]

(14)
where \( J_n \) is the Bessel function of first kind of order \( n \) and \( Y_n \) is the Bessel function of second kind of order \( n \), \((\alpha,a)^2\) are the roots of the equation 
\[
A(\alpha a)^4 - B(\alpha a)^2 + C = 0
\]
and the constant 
\[
a_i = \left(2 + \lambda^2\right)\sqrt{2^i + \Omega^2}, i = 1, 2
\]
Solving Equation 11, we get the solution for the symmetric mode as:
\[
\psi_n = [A_n J_n (\alpha_i a) + B_n Y_n (\alpha_i a)]\sin n\theta.
\] (15)
The solutions for the antisymmetric mode \( \tilde{\psi}_n \) and \( \tilde{\psi}_n \) are obtained from Equations 14 and 15 by replacing \( \sin n\theta \) by \( \cos n\theta \) and \( \cos n\theta \) by \( \sin n\theta \). If \((\alpha, a)^2 < 0 (i = 1, 2, 3)\), then the Bessel functions \( J_n \) and \( Y_n \) are to be replaced by the modified Bessel function \( I_n \) and \( K_n \) respectively. The integration constants \( A_n, B_n (i = 1, 2, 3) \) are to be determined from the boundary conditions.

**Boundary conditions and frequency equations**

In this paper, the free vibration of doubly connected polygonal (triangle, square, pentagon and hexagon) plate is considered. Since the polygonal cross-section of the boundary is irregular in shape, it is difficult to satisfy the boundary conditions along the outer and inner surface of the plate directly. Hence, in the same line of Nagaya (1981a, b, c, 1983a, b), the Fourier expansion collocation method is applied. Thus, the boundary conditions along the outer boundary of the plate are obtained as:
\[
(\sigma_{xx})_i = (\sigma_{yy})_i = (T)_i = 0
\] (16)
and for the inner boundary, the boundary conditions are:
\[
(\sigma_{xx}')_i = (\sigma_{yy}')_i = (T')_i = 0
\] (17)
where \( x \) is the coordinate normal to the boundary and \( y \) is the coordinate tangential to the boundary, \( \sigma_{xx}, \sigma_{yy}' \) are the normal stresses, \( \sigma_{xx}', \sigma_{yy} \) are the shearing stresses, \( T, T' \) are the thermal fields and \((...)_i \) is the value at the \( i \)-th segment of the outer and inner boundary respectively. Since the angle \( \gamma_i \) between the reference axis and the normal to \( i \)-th straight line boundary has a constant value in the segment as shown in Fig 1, we can obtain the transformed equations of the normal stress \( \sigma_{xx} \) and shearing stress \( \sigma_{yy} \) for \( i \)-th segment of the boundary are expressed as Nagaya (1983a) is:
\[
\begin{align*}
\sigma_{xx}' &= \lambda (u_{xx} + r^{-1}(u_{xx} + u_{yy})) + 2\mu(u_{xx}, \cos^2(\theta - \gamma) + r^{-1}(u_{xx} + u_{yy})\sin^2(\theta - \gamma)) \\
&+ 0.5(r^{-1}(u_{xx} - u_{yy}) - u_{yy})\sin 2(\theta - \gamma) - \beta T
\end{align*}
\] (18)
Substituting Equations 14 and 15 in Equations 16 and 17, and performing Fourier series expansion to the boundary, the boundary condition along the inner and outer surfaces are expanded in the form of double Fourier series. When the plate is symmetric about more than one axis, the boundary conditions in the case of symmetric mode can be written in the form of a matrix as given below:
\[
\begin{bmatrix}
E_{m1}^j & E_{m2}^j & \cdots & E_{mL}^j \\
F_{m1}^j & F_{m2}^j & \cdots & F_{mL}^j \\
G_{m1}^j & G_{m2}^j & \cdots & G_{mL}^j
\end{bmatrix}
\begin{bmatrix}
\epsilon_{m1}^j \\
\epsilon_{m2}^j \\
\vdots \\
\epsilon_{mL}^j
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
\] (19)
in which
\[
E_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m1}^j (R_i, \theta) \cos m \theta d \theta,
\]
\[
F_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m2}^j (R_i, \theta) \sin m \theta d \theta,
\]
\[
G_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m3}^j (R_i, \theta) \cos m \theta d \theta,
\]
\[
E_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m4}^j (R_i, \theta) \cos m \theta d \theta,
\]
\[
F_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m5}^j (R_i, \theta) \sin m \theta d \theta,
\]
\[
G_{m1}^j = (2\varepsilon_n / \pi) \sum_{i=1}^{L} \epsilon_{m6}^j (R_i, \theta) \cos m \theta d \theta
\] (20)
where \( j = 1, 2, 3, 4, 5 \) and \( \beta \), \( L \) is the number of segments,
$R_i$ is the coordinate $r$ at the inner boundary, $R_o$ is the coordinate $r$ at the outer boundary and $N$ is the number of truncation of the Fourier series. The coefficients $e_n^{(s)}$, $g_n^{(s)}$ are given in Appendix A.

GREEN – LINDSAY (GL) THEORY

The basic governing equations of motion and heat conduction in the absence of body force and heat source for a plate are considered from Sharma and Sharma (2002), in the context of GL theory is obtained by substituting $k = 2$ in the heat conduction equation (Equation 2), and in the stress-strain relation (Equation 3), thus we obtain:

$$K(T_{rr} + r^{-1} T_{r} + r^{-2} T_{rr}) - \rho c_v (\dot{T} + t_0) = T_0 \frac{\partial}{\partial t} \left[ \beta \left( u_r + r^{-1} (u_r + u_{\theta \theta}) \right) \right]$$

(21)

and

$$\sigma_{rr} = \lambda (e_r + e_{\theta \theta}) + 2 \mu e_r - \beta \left( T + t_0 \dot{T} \right)$$

$$\sigma_{\theta \theta} = \lambda (e_r + e_{\theta \theta}) + 2 \mu e_{\theta \theta} - \beta \left( T + t_0 \dot{T} \right)$$

(22)

Substituting Equations 4, 5 and 22 in Equations 1 along with Equation 21, we get the displacement equations of motion as follows:

$$(\lambda + 2 \mu) (u_r + r \dot{u}_r) - \rho \dot{u}_r + r^2 \left( \lambda + 3 \mu \right) u_{\theta \theta} - \beta (T + t_0) \dot{T} = \rho \ddot{u}_r$$

$$\mu (u_r + r \dot{u}_r) + r^2 \left( \lambda + 2 \mu \right) u_{\theta \theta} - \beta (T + t_0) \dot{T} = \rho \ddot{u}_r$$

$$K \left( T_{rr} + r^{-1} T_{r} + r^{-2} T_{rr} \right) - \rho c_v (\dot{T} + t_0) = T_0 \left[ \beta \left( u_r + r^{-1} (u_r + u_{\theta \theta}) \right) \right]$$

(23)

The frequency equations for GL theory are obtained similarly as discussed in the previous section solution of the problem. The coefficients $A$, $B$ and $C$ are given by:

$$A = iK \left( 2 + \lambda \right), \quad B = \left( \left( 2 + \lambda \right) \ddot{d} + i K \Omega^2 + \tau_1 \right), \quad C = \Omega^2 \ddot{d}.$$

(24)

where $A$, $B$ and $C$ are arbitrary constants. The Fourier coefficient $e_n^{(s)}, g_n^{(s)}$ are given in Appendix A, the term $a_i$ is to be replaced with $b_i$, where $b_i = \frac{\left( \left( 2 + \lambda \right) \ddot{V}^2 + \Omega^2 \right) \tau_1, i = 1,2,3,4,5,6}.$

COUPLED THEORY (CT) OF THERMOELASTICITY

The frequency equations for coupled theory of thermoelasticity are obtained by substituting the thermal relaxation times $t_0 = t_i = 0$ in the corresponding equation and solutions in the previous sections.

UNCOPLED THEORY (UCT) OF THERMOELASTICITY

The frequency equations for uncoupled theory of thermoelasticity are obtained by setting $\beta = K = c_v = T = T_0 = 0$ along with the thermoelastic coupling factors $t_0 = t_i = 0$ in the corresponding equations and solutions in the previous sections. Substituting $Y_n = Y_{n+1} = 0$ in the frequency equation, the resulting frequency equation of a solid plate matches with Equation 26 of Ponnusamy (2012).

RIGIDLY FIXED (CLAMPED) EDGE

The boundary conditions for rigidly fixed boundary is obtained by assuming that the displacements along the radial direction $u_r$, along the circumferential direction $u_\theta$ and the thermal field $T$ is equal to zero, thus we get the boundary conditions for the outer surface as:

$$(u_r)_{\theta} = (u_\theta)_{\theta} = (T) = 0$$

and the boundary condition for the inner surface of the plate as:

$$(u_r)_{\theta} = (u_\theta)_{\theta} = (T) = 0$$

(25)

Using Equation 9 in Equation 25, we can obtain the frequency equations for rigidly fixed boundary in the following form:

$$[b_{ij}] = 0, (i, j = 1,2,3,4,5,6). \quad \text{Where}$$

$$b_{ij} = \begin{cases} n J_n (\alpha_i \alpha_j), & i = 1, 2 \\ n J_n (\alpha_i \alpha_j), & i = 1, 2 \\ n J_n (\alpha_i \alpha_j), & i = 1, 2, \quad b_{33} = 0 \\ n J_n (\alpha_i \alpha_j), & i = 1, 2, \quad b_{33} = 0 \end{cases}$$

(26)

The remaining terms $b_{ij}, b_{12}, b_{14}, b_{13}, b_{53}, b_{63}, (j = 4,5,6)$ are
obtained by replacing \( J_\alpha \) and \( J_{\alpha+1} \) with \( Y_\alpha \) and \( Y_{\alpha+1} \)
respectively, and the constant \( d_i = \left[ \Omega^2 - (\alpha a)^2 \left( 2 + \frac{\lambda}{\mu} \right) \right] \).

**NUMERICAL RESULTS AND DISCUSSION**

The numerical analysis of the frequency equation is carried out for generalized thermoelastic doubly connected polygonal (square, triangle, pentagon and hexagon) plates, and the dimensions of each plate used in the numerical calculation are shown in Figure 2. The axis of symmetry is denoted by the lines in Figure 2. The material properties of copper at 42\( ^\circ \)K are taken from Erbay and Suhubi (1986) as Poisson ratio \( \nu = 0.3 \), density \( \rho = 8.96 \times 10^3 \text{ kg/m}^3 \), the Young’s modulus \( E = 2.139 \times 10^{11} \text{ N/m}^2 \), \( \lambda = 8.20 \times 10^{11} \text{ kg/m}^2 \cdot \text{s}^2 \), \( \mu = 4.20 \times 10^{10} \text{ kg/m}^2 \cdot \text{s}^2 \), \( c_v = 9.1 \times 10^{-2} \text{ m}^2 / \text{kg} \cdot \text{s}^2 \) and
\( K = 113 \times 10^{-2} \text{ kN/m} \) , and the thermal relaxation times considered from Sharma and Sharma (2002) as \( t_0 = 0.75 \times 10^{-13} \text{ sec} \), \( t_i = 0.5 \times 10^{-13} \text{ sec} \).

In the numerical calculation, the angle \( \theta \) is taken as an independent variable and the coordinate \( R_i \) and \( R_i \) are at the \( i \)-th segment of the boundary and is expressed in terms of \( \theta \). Substituting \( R_i \), \( R_i \) and the angle \( \gamma_i \), the distances between the reference axis and the normal to the \( i \)-th boundary line, the integrations of the Fourier coefficients \( e_i, f_i, g_i^i, \overline{e}_i, \overline{f}_i, \overline{g}_i \) can be expressed in terms of the angle \( \theta \). Using these coefficients in Equation 20, the frequencies are obtained for generalized thermoelastic polygonal plate.

**Polygonal cross-sectional plate**

The geometry of the ring shaped polygonal (triangle, square, pentagon and hexagon) plates is shown in Figure 2.

The geometrical relations for the polygonal doubly connected plates are given by Nagaya (1981b) as follows:

\[
R_i/a = \left[ \cos(\theta - \gamma_i) \right]^{-1} \]
\[
R_i/b = \left[ \cos(\theta - \gamma_i) \right]^{-1} \
\gamma_i = \gamma_i \quad (27) \]

where \( a = b + h \) and \( b \) is the apothem as shown in Figure 2, and \( h \) is the thickness of the plate. Here, the apothem \( b \) is taken as the reference length which is used to obtain the dimensionless expressions, and \( \gamma_i \) is the angle between the reference axis and the normal to the segment and is shown in Figure 1. In the present problem, two kinds of basic independent modes of wave propagation have been considered, namely, the longitudinal and flexural antisymmetric modes of vibrations.

**Longitudinal mode**

In longitudinal mode of square and hexagonal cross-section, the cross-section vibrates along the axis of the plate, so that the vibration displacements in the cross-sections are symmetrical about both the major and the minor axes. Hence, the frequency equations are obtained by choosing both the terms of \( m \) and \( n \) as 0, 2, 4, 6, ..., in Equation 19 for the numerical calculations. In the case of triangle and pentagonal shaped plate, the vibration and displacements are symmetrical about the major axis alone, hence the frequency equations are obtained from Equation 19 by choosing \( m \) and \( n \) as 0, 1, 2, 3, .... Since the boundary of the plate namely, triangle, square, pentagon and hexagon are irregular, it is difficult to satisfy the boundary conditions along the curved surface, and hence Fourier expansion collocation method is applied. That is the curved surface, in the range \( \theta = 0 \) and \( \theta = \pi \) is divided into 20 segments, such that the distance between any two segments is negligible and the integrations are performed for each segment numerically by using the Gauss five point formula. The non-dimensional frequencies are computed for \( 0 < \Omega \leq 1.0 \), using the secant method [applicable for the complex roots, (Antia, 2002)].

The geometric relation for the polygonal plate is given in Equation 27, which is used for the numerical calculation. The non-dimensional frequencies of longitudinal and flexural antisymmetric modes are plotted in the form of dispersion curves and are shown in Figures 3 to 9. The notations used in the figures namely ICOF, IFOC and ICOC respectively denotes the Inner Clamped and Outer Clamped edges, Inner Free and Outer Free edges, Inner Free and Outer Clamped edges and Inner Clamped and Outer Clamped edges.

**Flexural mode**

In flexural mode of square and hexagonal cross-section, the vibration and displacements are anti-symmetrical about the major axis and symmetrical about the minor axis. Hence, the frequency equation is obtained from Equation 19 by changing \( \cos n\theta \) by \( \sin n\theta \) and \( \sin n\theta \) by \( \cos n\theta \) and choosing \( n, m = 1, 3, 5, 7,... \). In the case of
Figure 1. Geometry of a straight line segment.

\[
\begin{align*}
\theta_0 &= 0^\circ \quad \gamma_1 = 60^\circ \\
\theta_1 &= 120^\circ \quad \gamma_2 = 180^\circ \\
\theta_2 &= 180^\circ \quad I = 2 \\
\end{align*}
\]

(a)

Figure 2. Geometry of ring shaped polygonal plates. (a) triangle, (b) square, (c) pentagon, (d) hexagon.

\[
\begin{align*}
\theta_0 &= 0^\circ \quad \gamma_1 = 45^\circ \\
\theta_1 &= 90^\circ \quad \gamma_2 = 135^\circ \\
\theta_2 &= 180^\circ \quad I = 2 \\
\end{align*}
\]

(b)

\[
\begin{align*}
\theta_0 &= 0^\circ \quad \gamma_1 = 0^\circ \\
\theta_1 &= 36^\circ \quad \gamma_2 = 72^\circ \\
\theta_2 &= 108^\circ \quad \gamma_3 = 144^\circ \\
\theta_3 &= 180^\circ \quad I = 3 \\
\end{align*}
\]

(c)

\[
\begin{align*}
\theta_0 &= 0^\circ \quad \gamma_1 = 30^\circ \\
\theta_1 &= 60^\circ \quad \gamma_2 = 72^\circ \\
\theta_2 &= 120^\circ \quad \gamma_3 = 150^\circ \\
\theta_3 &= 180^\circ \quad I = 3 \\
\end{align*}
\]

(d)
triangle and pentagonal plate, the vibration and displacements are anti-symmetrical about the minor axis, hence the frequency equations are obtained by choosing $n,m = 1,2,3,\ldots$.

The frequency equation of a thick polygonal plate without thermal field is chosen to compare the present results with the results of Nagaya (1981b). The non-dimensional frequencies $\Omega$ are calculated for a ring shaped polygonal plates with ICOC edges. The frequencies obtained by the present method matches with the results of Nagaya (1981b) and are shown in Table 1. Hence, the analysis is extended to understand the characteristics of vibration of the plate with thermal field. The notations namely, $S_1, S_2, S_3, S_4, S_5$ and $A_1, A_2, A_3, A_4, A_5$ used in the tables respectively represent the symmetric and antisymmetric modes of vibration, and the subscripts 1, 2, 3, 4, 5 represent the first, second, third, fourth and fifth modes of vibrations.

A comparison is made between the frequency response of different modes of vibration for GL, LS, CT and UCT theories of thermoelasticity for the longitudinal modes of triangular and pentagonal shaped plates and is presented in Table 2, square and hexagonal shaped plates and is presented in Table 3 and the frequency responses of the UCT of thermo elasticity for longitudinal modes of triangle, square, pentagon and hexagonal cross-sectional plates are presented in Table 4. From Tables 2, 3 and 4, it is observed that, as the modes of vibration increases, the non-dimensional frequencies also increase in the LS, GL, CT and UCT theories of thermoelasticity. The frequencies of LS, GL, CT theories of thermoelasticity for the flexural antisymmetric modes of vibrations of triangle and pentagon and square and hexagon are compared respectively and are shown in Tables 5 and 6. Also, the frequency responses of the UCT of thermoelasticity for flexural antisymmetric modes of triangle, square, pentagon and hexagonal shaped plates are presented in Table 7. From Tables 5, 6 and 7, it is observed that as the mode of vibration increase, the non-dimensional

### Table 1. Comparison between the non-dimensional frequencies $\Omega$ of present results with the results of Nagaya (1981b) of a ring shaped polygonal plates with clamped outside and clamped inside (ICOC) edges.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>Mode</th>
<th>Triangle</th>
<th>Square</th>
<th>Pentagon</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Nagaya</td>
<td>Author</td>
<td>Nagaya</td>
<td>Author</td>
</tr>
<tr>
<td>0.1</td>
<td>S1</td>
<td>4.148</td>
<td>4.146</td>
<td>4.850</td>
<td>4.849</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>4.392</td>
<td>4.393</td>
<td>4.997</td>
<td>4.997</td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>4.547</td>
<td>4.551</td>
<td>5.889</td>
<td>5.886</td>
</tr>
<tr>
<td>0.15</td>
<td>S1</td>
<td>4.269</td>
<td>4.267</td>
<td>5.128</td>
<td>5.127</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>4.510</td>
<td>4.511</td>
<td>5.271</td>
<td>5.272</td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>4.769</td>
<td>4.765</td>
<td>6.069</td>
<td>6.068</td>
</tr>
<tr>
<td>0.2</td>
<td>S1</td>
<td>4.413</td>
<td>4.412</td>
<td>5.431</td>
<td>5.431</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>4.622</td>
<td>4.613</td>
<td>5.573</td>
<td>5.572</td>
</tr>
<tr>
<td>0.25</td>
<td>S1</td>
<td>4.474</td>
<td>4.573</td>
<td>5.757</td>
<td>5.756</td>
</tr>
</tbody>
</table>

### Table 2. Comparison between the non-dimensional frequencies $\Omega$ of Lord-Shulman (LS) Green-Lindsay (GL) and Coupled Theory (CT) theories of thermoelasticity for longitudinal modes of Triangular and Pentagonal cross-sectional plates for the aspect ratio $a/b = 1.2$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Triangle</th>
<th>Pentagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LS</td>
<td>GL</td>
</tr>
<tr>
<td>S1</td>
<td>4.1151</td>
<td>4.1403</td>
</tr>
<tr>
<td>S2</td>
<td>4.8474</td>
<td>4.9872</td>
</tr>
<tr>
<td>S3</td>
<td>5.5647</td>
<td>5.6966</td>
</tr>
</tbody>
</table>
Table 3. Comparison between the non-dimensional frequencies $\Omega$ of Lord-Shulman (LS) Green-Lindsay (GL) and Coupled Theory (CT) theories of thermoelasticity for longitudinal modes of Square and Hexagonal cross-sectional plates for the aspect ratio $a/b = 1.2$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Square</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LS</td>
<td>GL</td>
</tr>
<tr>
<td>S1</td>
<td>4.2469</td>
<td>4.1053</td>
</tr>
<tr>
<td>S2</td>
<td>4.9545</td>
<td>4.9501</td>
</tr>
<tr>
<td>S3</td>
<td>5.6625</td>
<td>5.6556</td>
</tr>
<tr>
<td>S5</td>
<td>7.0732</td>
<td>6.9290</td>
</tr>
</tbody>
</table>

Table 4. Comparison between the non-dimensional frequencies $\Omega$ of longitudinal modes of triangle, square, pentagon and hexagonal cross-sectional uncoupled theory (UCT) of thermoelastic plate for the aspect ratio $a/b = 1.2$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Triangle</th>
<th>Square</th>
<th>Pentagon</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LS</td>
<td>GL</td>
<td>CT</td>
<td>LS</td>
</tr>
<tr>
<td>S1</td>
<td>4.1416</td>
<td>4.2469</td>
<td>4.0787</td>
<td>4.2400</td>
</tr>
<tr>
<td>S2</td>
<td>4.8473</td>
<td>4.9544</td>
<td>4.7833</td>
<td>4.9552</td>
</tr>
<tr>
<td>S3</td>
<td>5.5544</td>
<td>5.6648</td>
<td>5.4945</td>
<td>5.6591</td>
</tr>
<tr>
<td>S5</td>
<td>6.9694</td>
<td>7.2190</td>
<td>6.9006</td>
<td>7.0781</td>
</tr>
</tbody>
</table>

Table 5. Comparison between the non-dimensional frequencies $\Omega$ of the Lord-Shulman (LS) Green-Lindsay (GL) and Coupled Theory (CT) theories of thermoelasticity for flexural antisymmetric modes of Triangular and Pentagonal cross-sectional plates for the aspect ratio $a/b = 1.2$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Triangle</th>
<th>Pentagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LS</td>
<td>GL</td>
</tr>
<tr>
<td>A1</td>
<td>4.1440</td>
<td>4.1536</td>
</tr>
<tr>
<td>A2</td>
<td>4.8602</td>
<td>4.8621</td>
</tr>
<tr>
<td>A3</td>
<td>5.5753</td>
<td>5.5712</td>
</tr>
<tr>
<td>A4</td>
<td>6.2691</td>
<td>6.2785</td>
</tr>
<tr>
<td>A5</td>
<td>6.9696</td>
<td>6.9857</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Square</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LS</td>
<td>GL</td>
</tr>
<tr>
<td>A1</td>
<td>4.2377</td>
<td>4.1785</td>
</tr>
<tr>
<td>A2</td>
<td>4.9598</td>
<td>4.8095</td>
</tr>
<tr>
<td>A3</td>
<td>5.6645</td>
<td>5.5171</td>
</tr>
<tr>
<td>A4</td>
<td>6.3672</td>
<td>6.2246</td>
</tr>
<tr>
<td>A5</td>
<td>7.0761</td>
<td>6.9380</td>
</tr>
</tbody>
</table>

The non-dimensional frequency $\Omega$ of a ring shaped polygonal plate is obtained for various combinations of

frequency also increases in triangular, square, pentagonal and hexagonal shaped plates.
Table 7. Comparison between the non-dimensional frequencies $|\Omega|$ of flexural antisymmetric modes of triangle, square, pentagon and hexagonal cross-sectional uncoupled theory (UCT) of thermoelastic plate for the aspect ratio $a/b = 1.2$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Triangle</th>
<th>Square</th>
<th>Pentagon</th>
<th>Hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>4.1537</td>
<td>4.1026</td>
<td>3.9578</td>
<td>4.2375</td>
</tr>
<tr>
<td>A2</td>
<td>4.8603</td>
<td>4.8083</td>
<td>4.8025</td>
<td>4.9551</td>
</tr>
<tr>
<td>A3</td>
<td>5.5675</td>
<td>5.5209</td>
<td>5.5011</td>
<td>5.6616</td>
</tr>
<tr>
<td>A4</td>
<td>6.2752</td>
<td>6.2399</td>
<td>6.2069</td>
<td>6.3684</td>
</tr>
<tr>
<td>A5</td>
<td>6.9831</td>
<td>6.9355</td>
<td>6.9138</td>
<td>7.0787</td>
</tr>
</tbody>
</table>

Figure 3. Aspect ratio $a/b = 0.0, 0.1, 0.5, 1.0$ versus non-dimensional frequency $|\Omega|$ for Lord-Shulman (LS) theory of thermoelasticity for longitudinal modes of triangular cross-sectional plate with ICOF edges.

the outer and inner boundary conditions. To clarify the effect of the hole, the dispersion curves are drawn between the aspect ratio $a/b$ versus the frequency $|\Omega|$. The solution for a doubly connected plate when aspect ratio $a/b$ reduces to zero does not coincide with the solid plate, the fundamental frequency should be little lower than the case of solid plate, because the rigidity of the plate has decreased. This behavior is observed in all the graphs of longitudinal and flexural antisymmetric modes of vibrations. From all the graphs, it can be noted that at first the frequency decreases and then it starts to increase with $a/b$ when the plate has a hole. The effect of frequencies of the inner boundary with clamped or simply supported is large compared with that of free edge boundary conditions.

Graphs are drawn for ICOF and IFOC edge boundary conditions for LS theory of thermoelasticity between the non-dimensional aspect ratio $a/b = 0.0, 0.1, 0.5, 1.0$ versus non-dimensional frequency $|\Omega|$ for longitudinal modes of triangular plates respectively and are shown in Figures 3 and 4. From Figures 3 and 4, it is observed that the non-dimensional frequency $|\Omega|$ increases with respect to its aspect ratio $a/b$, also it is noted that the frequency for
Figure 4. Aspect ratio $a/b = 0.0, 0.1, 0.5, 1.0$ versus non-dimensional frequency $|\Omega|$ for Lord-Shulman (LS) theory of thermoelasticity for longitudinal modes of triangular cross-sectional plate with IFOC edges.

ICOF surface have higher frequency than the frequency of IFOC edge boundary conditions. The frequencies increase for higher modes of vibrations, and the cross over points in the trend line indicates the transfer of heat energy between the modes of vibrations. The transfer of heat energy is higher in the lower modes of vibrations as compared to the higher modes.

A comparison is made between ICOF, IFOC and ICOC edge boundary conditions of LS theory of thermoelasticity for longitudinal and flexural antisymmetric modes of vibrations of pentagonal cross section respectively and is shown in Figure 5 and 6. From Figures 5 and 6, it is observed that the frequency is higher for a plate with both edges clamped as comparing with the other type of boundary conditions namely, ICOF and IFOC. The same physical behavior of a plate is obtained for longitudinal modes of square which is shown in Figure 7 for GL theory of thermoelasticity. From Figure 7, it is observed that when the aspect ratio $a/b = 0.0$, the dispersion curves of ICOF and IFOC are linear but for the case ICOC the curve increases greatly due to the increase in temperature. Similarly, when the aspect ratio $a/b = 1.0$ the dispersion curve of ICOF is linear but the dispersion curves of IFOC and ICOC first increases then decreases due to the increase in temperature. A dispersion curves are drawn for triangle, square, pentagon and hexagonal plates with ICOF edges of GL theory of thermoelasticity for the aspect ratio $a/b = 0.5$ versus non-dimensional frequency of longitudinal and flexural antisymmetric modes of vibrations and are shown in Figures 8 and 9, respectively. From Figures 8 and 9, it is observed that the dispersions behavior of triangle and pentagon are same. Similarly the dispersion curve of square and hexagon are in similar pattern. The crossover point denotes the transfer of heat energy between the modes of vibrations. From Figure 8, it is also observed that the dispersion curves of square and hexagon contact each other at some point. This cross over point represents the transfer of heat energy between modes of vibration of square and hexagon.

CONCLUSIONS

Free vibration analysis of generalized thermoelastic doubly connected polygonal shaped plate is studied using the Fourier expansion collocation method. The equation of motion based on two-dimensional theory of elasticity is applied under the plane strain assumption of generalized thermoelastic plate of polygonal shape composed of homogeneous isotropic material. The frequency equations are obtained by satisfying the boundary conditions along the inner and outer surface of
Figure 5. Comparison between ICOF, IFOC and ICOC edges for Lord-Shulman (LS) theory of thermoelasticity for the aspect ratio $a/b = 0.0, 1.0$ versus non-dimensional frequency $|\Omega|$ of longitudinal modes of pentagonal cross-sectional plate.

Figure 6. Comparison between ICOF, IFOC and ICOC edges for Lord-Shulman (LS) theory of thermoelasticity for the aspect ratio $a/b = 0.0, 1.0$ versus non-dimensional frequency $|\Omega|$ for flexural antisymmetric modes of pentagonal cross-sectional plate.
Figure 7. Comparison between ICOF, IFOC and ICOC edges for Green-Lindsay (GL) theory of thermoelasticity for the Aspect ratio \(a/b = 0.0, 1.0\) versus Non-dimensional frequency \(|\Omega|\) of longitudinal modes of square cross-sectional plate.

Figure 8. Dispersion curves of triangle, square, pentagon and hexagonal shaped plate with ICOF edges of Green-Lindsay (GL) theory of thermoelasticity for the Aspect ratio \(a/b = 0.5\) versus Non-dimensional frequency \(|\Omega|\) of longitudinal modes.
the polygonal plate. The numerical calculations are carried out for triangular, square, pentagonal and hexagonal shaped plates. The computed non-dimensional frequencies are compared with the Lord-Shulman (LS), Green-Lindsay (GL), coupled theory (CT) and uncoupled theory (UCT) theories of thermoelasticity and they are presented in the tables. The dispersion curves are drawn for longitudinal and flexural antisymmetric modes of vibration.

**REFERENCES**


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APPENDIX A

The expressions \( e'_n \) used in Equation 20 are given as follows:

\[
e'_n = 2 \left\{ n(n-1)J_n(\alpha, ax) + (\alpha, ax)J_{n+1}(\alpha, ax) \right\} \cos 2(\theta - \gamma) \cos n\theta \\
- \ x^2 \left\{ (\alpha, ax)^2 + \left[ \alpha^2 + 2 \cos^2(\theta - \gamma) \right] \right\} J_n(\alpha, ax) \cos n\theta \\
+ 2n \left\{ (n-1)J_n(\alpha, ax) - (\alpha, ax)J_{n+1}(\alpha, ax) \right\} \sin n\theta \sin 2(\theta - \gamma), \ i = 1,2
\]  
(A1)

\[
e''_n = 2 \left\{ n(n-1)J_n(\alpha, ax) - (\alpha, ax)J_{n+1}(\alpha, ax) \right\} \cos n\theta \cos 2(\theta - \gamma) \\
+ 2 \left\{ (n-1)(\alpha, ax)^2 J_n(\alpha, ax) + (\alpha, ax)J_{n+1}(\alpha, ax) \right\} \sin n\theta \sin 2(\theta - \gamma)
\]  
(A2)

\[
e''_n = 2 \left\{ n(n-1)Y_n(\alpha, ax) - (\alpha, ax)Y_{n+1}(\alpha, ax) \right\} \cos n\theta \cos 2(\theta - \gamma) \\
- x^2 \left\{ (\alpha, ax)^2 + \left[ \alpha^2 + 2 \cos^2(\theta - \gamma) \right] \right\} Y_n(\alpha, ax) \cos n\theta \\
+ 2n \left\{ (n-1)Y_n(\alpha, ax) - (\alpha, ax)Y_{n+1}(\alpha, ax) \right\} \sin n\theta \sin 2(\theta - \gamma), \ i = 5,6
\]  
(A3)

\[
f'_n = 2 \left\{ n(n-1)- (\alpha, ax)^2 \right\} J_n(\alpha, ax) + (\alpha, ax)J_{n+1}(\alpha, ax) \} \cos n\theta \sin 2(\theta - \gamma) \\
+ 2 \left\{ n(n-1)- (\alpha, ax)^2 \right\} J_n(\alpha, ax) - (n-1)J_n(\alpha, ax) \} \sin n\theta \cos 2(\theta - \gamma), i = 1,2
\]  
(A5)

\[
f'_n = 2 \left\{ n(n-1)- (\alpha, ax)^2 \right\} J_n(\alpha, ax) - (\alpha, ax)J_{n+1}(\alpha, ax) \} \cos n\theta \sin 2(\theta - \gamma) \\
- 2 \left\{ (\alpha, ax)J_{n+1}(\alpha, ax) - (\alpha, ax)J_n(\alpha, ax) \right\} \cos n\theta \cos 2(\theta - \gamma)
\]  
(A6)

\[
f''_n = 2 \left\{ n(n-1)- (\alpha, ax)^2 \right\} Y_n(\alpha, ax) - (\alpha, ax)Y_{n+1}(\alpha, ax) \} \cos n\theta \sin 2(\theta - \gamma) \\
- 2 \left\{ (\alpha, ax)Y_{n+1}(\alpha, ax) - (\alpha, ax)Y_n(\alpha, ax) \right\} \cos n\theta \cos 2(\theta - \gamma)
\]  
(A7)

\[
f''_n = 2 \left\{ n(n-1)- (\alpha, ax)^2 \right\} Y_n(\alpha, ax) + (\alpha, ax)Y_{n+1}(\alpha, ax) \} \cos n\theta \sin 2(\theta - \gamma) \\
+ 2 \left\{ (\alpha, ax)Y_{n+1}(\alpha, ax) - (\alpha, ax)Y_n(\alpha, ax) \right\} \sin n\theta \cos 2(\theta - \gamma), i = 5,6
\]  
(A8)

\[
k'_n = d_n \left[ n \cos(n\theta + \gamma)J_n(\alpha, ax) - (\alpha, ax)J_{n+1}(\alpha, ax) \cos(\theta - \gamma) \cos n\theta \right], i = 1,2
\]  
(A9)

\[ k''_1 = 0.0, \ k''_2 = 0.0 \quad (A10)
\]

\[
k'_n = d_n \left[ n \cos(n\theta + \gamma)Y_n(\alpha, ax) - (\alpha, ax)Y_{n+1}(\alpha, ax) \cos(\theta - \gamma) \cos n\theta \right], i = 5,6
\]  
(A11)

The expressions \( e'_n \) and \( g'_n \) are obtained by replacing \( \cos n\theta \) by \( \sin n\theta \) and \( \sin n\theta \) by \( \cos n\theta \) in the Equations A1 to A11.