**Advancement in Scientific and Engineering Research**
**Vol. 2(2), pp. 18-22, June 2014**
**ISSN: 2384-7336**
**Short Communication**

# Numerical solution of second order linear and non linear integro-differential equations by cubic spline collocation method

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Accepted 20th May, 2014

***Abstract.*** In this paper, spline collocation methods are applied to solve linear and nonlinear Fredholm – integro-differential equations. Two spline collocation methods are presented in the paper and examples are used to illustrate the ability of the spline collocation methods. The results reveal that the proposed spline methods are very effective and simple and results obtained are compared favorably with known results in closed form solutions.

**Keywords:** Integro-differential equation, standard spline approximation, non-polynomial spline approximation, collocation.

## INTRODUCTION

Integro – differential equations that are considered in this work are classified into Fredholm – integro differential and Volterra – integro-differential equations.

### Fredholm – integro – differential equations

The second order linear and non linear Fredholm – integro – differential equations that are considered in this work are defined as follows:

$$u''(x) + \int_a^b W(x,t)\,u(t)\,dt = g(x) \qquad (1)$$

and

$$u''(x) + \int_a^b W(x,t)\,F(u(t))\,dt = h(t) \qquad (2)$$

*with*

$$u(a) = u_0,\ u'(a) = u_1 \qquad (3)$$

Where $u(x)$ is the unknown function to be determined, $F(x), g(x), h(x)$ and $W(x,t)$ are given smooth continuous functions, $a$ and $b$ are constants and F(x) is general, a nonlinear. Equation 1 is referred to as linear Fredholm – integro – differential equation, while Equation 2 is the non linear type.

### Volterra – integro – differential equations

The second order Volterra – integro – differential equations that is considered in this work is defined as:

$$u''(x) + \int_a^x W(x,t)\,u(t)\,dt = g(x) \qquad (4)$$

and

$$u''(x) + \int_a^x W(x,t)\,F(u(t))\,dt = h(t) \qquad (5)$$

with

$$u(a) = u_0,\ u(a) = u_1 \qquad (6)$$

Where $u(x)$ is the unknown function to be determined, $F(x), g(x), h(x)$ and $W(x,t)$ are given smooth continuous functions, $a$ and $b$ are constants and F(x) is general, a nonlinear.

Equation 4 is referred to as the linear Volterra – integro differential equation, while Equation 5 is the non linear type.

Integro-differential equations are of significant importance in modeling numerous physical processes such as signal processing and neural networks (Davis, 1962; Kanwal, 1997; Micula and Pavel, 1992.

In recent years, many analytical and numerical methods have been proposed by various authors to solve integro–differential equations (Linz, 1985). Among such numerical methods are Homotopy Analysis Methods (Awadeh et al., 2004), Iterative Methods (Laidlaw and Phillips, 1972), Orthogonal Polynomial Random Matrix model 0f $N \times N$ (Craig and Harold, 1994). The use of non smooth initial value to solve integro–differential equations, was developed (Karkarashvili, 1993; Golberg, (1973).

During the last five decades, there has been growing interest in developing and using highly accurate numerical methods based on spline approximation for the solution of both linear and non linear integro–differential equation (Ahlberg et al., 1976; Joseph and Gene, 1973; Sastry, 2000).

Polynomial spline collocation method has been studied (Brumer and Tang, 1989; Vilmos, 1999). The B – spline method has been considered by Hesan-Elden and Hossien (2012). The cubic spline approximation for solving ordinary differential, partial differential and integral equations has been proposed (Gegele, 2004; Ogunlaran, 2012; Ogunlaran and Taiwo, 2013; Sastry, (1976).

The technique that we used is the standard spline collocation and non polynomial spline based on the combination of low degree polynomials and trigonometric functions.

## STANDARD SPLINE COLLOCATION METHOD

Here, we modified standard spline approximation developed in Gegele (2004) to solve second order linear and non linear Fredholm – integro – differential equations.

### Linear second order Fredholm - integro differential equation

The spline approximation defined in the interval [a, b] such that $a = x_0 < x_1 < ... < x_N = b$ is given as:

$$s_j = \frac{(x_j - x)^3}{6h} M_{j-1} + \frac{(x - x_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(x_j - x) + \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(x - x_{j-1}), \ j = 1, 2, ...N \quad (7)$$

and

$$s_{j+1} = \frac{(x_{j+1} - x)^3}{6h} M_j + \frac{(x - x_j)^3}{6h} M_{j+1} + \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(x_{j+1} - x) + \frac{1}{h}\left(u_{j+1} - \frac{h^2}{6} M_{j+1}\right)(x - x_j), \ j = 0, 1, ...N-1 \quad (8)$$

Where $M_j = s''(x_j)$, $u_j = u(x_j)$

Putting Equation 8 in Equation 1 yields,

$$M_j = \int_a^b W(x,t) \left\{ \frac{(t_j - t)^3}{6h} M_{j-1} + \frac{(t - t_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(t_j - t) + \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(t - t_{j-1}) \right\} dt = g(x),$$
$$j = 1, 2, ..., N \quad (9)$$

Now, we approximate Equation 9 to get

$$M_j + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} W(x,t) \left\{ \frac{(t_j - t)^3}{6h} M_{j-1} + \frac{(t - t_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(t_j - t) + \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(t - t_{j-1}) \right\} dt = g(x) \quad (10)$$

Setting $t = t_{j-1} + qh, \quad j = 1, 2, ..., N$.

Collocating Equation 10 at point $x_k = a + \frac{(b-a)}{N} k, \ k = 1, 2, ..., N-1$, after simplification becomes

$$M_j + \sum_{j=1}^{N} \int_0^1 W(x_k,t) \left\{ \frac{(t_j - t)^3}{6h} M_{j-1} + \frac{(t - t_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(t_j - t) + \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(t - t_{j-1}) \right\} dt = g(x_k) \quad (10)$$

Similarly, Equation 8 becomes

$$M_{j+1} + h\sum_{j=1}^{N} \int_0^1 W(x_k, t_j + qh) \left\{ \frac{(1-q)^3 h^2}{6} M_j + \frac{q^3 h^2}{6} M_{j+1} + \left(u_j - \frac{h^2}{6} M_j\right)(1-q) + \left(u_{j+1} - \frac{h^2}{6} M_{j+1}\right) q \right\} dq = g(x_k) \quad (12)$$

Equations 11 and 12 together with Equation 3 form a set of (N+ 2) algebraic linear equation in (N+2) unknowns.

### Nonlinear second order Fredholm – integro differential equations

The spline approximation given in Equation 7 is substituted into Equation 2 to give:

$$M_j + \int_a^b W(x,t) F \left\{ \begin{array}{l} \frac{(t_j - t)^3}{6h} M_{j-1} + \frac{(t - t_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(t_j - t) \\ + \frac{1}{h}\left(u_j - \frac{h^2}{h} M_j\right)(t - t_{j-1}) \end{array} \right\} dt = h(x) \quad (13)$$

Now, we approximate Equation 13 to get

$$M_j + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} W(x,t) F \left\{ \begin{array}{l} \frac{(t_j - t)^3}{6h} M_{j-1} + \frac{(t - t_{j-1})^3}{6h} M_j + \frac{1}{h}\left(u_{j-1} - \frac{h^2}{6} M_{j-1}\right)(t_j - t) + \\ \frac{1}{h}\left(u_j - \frac{h^2}{6} M_j\right)(t - t_{j-1}) \end{array} \right\} dt = h(x) \quad (14)$$

*Setting* $t = t_{j-1} + qh$ and collocating at

$x_k = a + \frac{(b-a)}{N}k$, k = 0,1,..., N to obtain

$$M_j + h\sum_{j=1}^{N}\int_0^1 W(x_k, t_{j-1}+qh)F\left\{\begin{array}{l}\frac{(1-q)^3h^2}{6}M_{j-1}+\frac{q^3h^2}{6}M_j+\left(u_{j-1}-\frac{h^2}{6}M_{j-1}\right)(1-q)\\ +\left(u_j-\frac{h^2}{6}M_j\right)(q)\end{array}\right\}dq = h(x_k) \quad (15$$

Similarly, Equation 8 becomes

$$M_{j+1} + h\sum_{j=1}^{N}\int_0^1 W(x_k, t_j+qh)\,F\left\{\begin{array}{l}\frac{(1-q)^3h^2}{6}M_j+\frac{q^3h^2}{6}M_{j+1}+\left(u_j-\frac{h^2}{6}M_j\right)(1-q)\\ +\left(u_{j+1}-\frac{h^2}{6}M_{j+1}\right)q\end{array}\right\}dq = h(x_k) \quad (16)$$

Equations 15 and 16 together with (3) form set of (N+2) algebraic linear equations in (N+2) unknowns.

## NON POLYNOMIAL SPLINE COLLOCATION METHOD

Here, we applied the non polynomial Spline function based on low degree polynomial and trigonometric function. The Spline approximation defined in the interval [a, b] such that

$a=x_0 < x_1 < \ldots < x_N = b$ is given as:

$$s_j(x) = a_j + b_j(x-x_j) + c_j\sin\tau(x-x_j) + d_j\cos\tau(x-x_j) \quad (17)$$

Where $a_j$, $b_j$, $c_j$, $d_j$ are constants to be determined.

Differentiating Equation 17 twice and after simplification, we obtain:

$$s_j(x) = u_j + \frac{M_j}{\tau^2} + \left\{\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right\}(x-x_j) + \frac{1}{\tau^2\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(x-x_j)$$
$$-\frac{M_j}{\tau^2}\cos\tau(x-x_j) \qquad j=0,1,\ldots,N-1 \quad (18)$$

Where $M_j = s''(x_j)$, $u_j = u(x_j)$, $\tau$ is a free parameter and $\theta = \tau h$

Similarly,

$$s_{j-1}(x) = u_{j-1} + \frac{M_{j-1}}{\tau^2} + \left\{\frac{1}{h}(u_j-u_{j-1}) + \frac{1}{\tau\theta}(M_j-M_{j-1})\right\}(x-x_j) + \frac{1}{\tau^2\sin\theta}(M_{j-1}\cos\theta - M_j)\sin\tau(x-x_j)$$
$$-\frac{M_{j+1}}{\tau^2}\cos\tau(x-x_j), \; j=1,2,\ldots N \quad (19)$$

3.1

### Linear second order Fredholm - integro differential equation

Putting Equation 17 in Equation 1 yields:

$$M_j + \int_a^b W(x,t)\left\{u_j + \frac{M_j}{\tau} + \left[\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right](t-t_j) + \frac{1}{\tau\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(t-t_j)\right.$$
$$\left.-\frac{M_{j+1}}{\tau^2}\cos\tau(t-t_j)\right\}dt = g(x) \quad (20)$$

Now, we approximate Equation 20 to get:

$$M_j + \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} W(x,t)\left\{u_j + \frac{M_j}{\tau^2} + \left[\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right](t-t_j) + \frac{1}{\tau^2\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(t-t_j)\right.$$
$$\left.-\frac{M_j}{\tau^2}\cos\tau(t-t_j)\right\}dt = g(x) \quad (21)$$

*Setting* $t = t_j + qh$ and collocating at $x_k = a + \frac{(b-a)}{N}k$, $k = 0,1,\ldots, N$ to obtain

$$M_j + \sum_{j=1}^{N}\int_0^1 W(x_k, t_j+qh)\left\{u_j + \frac{M_j}{\tau} + \left[\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right]qh + \frac{1}{\tau^2\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(qh)\right.$$
$$\left.-\frac{M_j}{\tau^2}\cos\tau(qh)\right\}dq = g(x_k), \quad j=0,1,\ldots,N+1 \quad (22)$$

Similarly, Equation 21 becomes:

$$M_{j-1} + \sum_{j=1}^{N}\int_0^1 W(x_k, t_{j-1}+qh)\left\{u_{j-1} + \frac{M_{j-1}}{\tau^2} + \left[\frac{1}{h}(u_j-u_{j-1}) + \frac{1}{\tau\theta}(M_j-M_{j-1})\right]qh + \frac{1}{\tau^2\sin\theta}(M_{j-1}\cos\theta - M_j)\sin\tau(qh)\right.$$
$$\left.-\frac{M_{j-1}}{\tau^2}\cos\tau(qh)\right\}dq = g(x_k), \; j=1,2,\ldots,N \quad (23)$$

Equations 22 and 23 together with Equation 3 form set of (N+2) algebraic linear equations in (N+2) unknowns.

### Non linear Fredholm – Integro differential equations

Putting Equation 17 in Equation 2, we have:

$$M_j + \int_a^b W(x,t)\,F\left\{u_j + \frac{M_j}{\tau} + \left[\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right](t-t_j) + \frac{1}{\tau\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(t-t_j)\right.$$
$$\left.-\frac{M_{j+1}}{\tau^2}\cos\tau(t-t_j)\right\}dt = h(x) \quad (24)$$

Now, we

approximate Equation 24 to get

$$M_j + \sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} W(x,t)\,F\left\{u_j + \frac{M_j}{\tau^2} + \left[\frac{1}{h}(u_{j+1}-u_j) + \frac{1}{\tau\theta}(M_{j+1}-M_j)\right](t-t_j) + \frac{1}{\tau^2\sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(t-t_j)\right.$$
$$\left.-\frac{M_j}{\tau^2}\cos\tau(t-t_j)\right\}dt = h(x) \quad (25)$$

*Setting* $t = t_j + qh$ and collocating at

**Table 1.** Numerical Results for Example 1.

| X | Exact solution | Proposed method | Error |
|---|---|---|---|
| 0 | 1.000000000 | 1.000000000 | 0.000000000 |
| 0.1 | 1.106666666 | 1.106667015 | $3.489\times10^{-6}$ |
| 0.2 | 1.220000000 | 1.220000034 | $3.410\times10^{-6}$ |
| 0.3 | 1.330000000 | 1.330002983 | $2.983\times10^{-6}$ |
| 0.4 | 1.426666666 | 1.426669503 | $2.837\times10^{-6}$ |
| 0.5 | 1.500000000 | 1.500002602 | $2.602\times10^{-6}$ |
| 0.6 | 1.540000000 | 1.540002591 | $2.591\times10^{-6}$ |
| 0.7 | 1.536666666 | 1.536669095 | $2.429\times10^{-6}$ |
| 0.8 | 1.480000000 | 1.480001994 | $1.994\times10^{-6}$ |
| 0.9 | 1.360000000 | 1.360001405 | $1.405\times10^{-6}$ |
| 1.0 | 1.166666666 | 1.666666770 | $1.067\times10^{-8}$ |

**Table 2.** Numerical Results for Example 2.

| X | Exact solution | Proposed method | Error |
|---|---|---|---|
| 0 | 1.000 | 1.000000000 | 1.000000000 |
| 0.1 | 1.049 | 1.049000060 | $6.008\times10^{-8}$ |
| 0.2 | 1.192 | 1.192000079 | $7.918\times10^{-8}$ |
| 0.3 | 1.423 | 1.423000843 | $8.432\times10^{-7}$ |
| 0.4 | 1.736 | 1.736000688 | $6.884\times10^{-7}$ |
| 0.5 | 2.125 | 2.125000572 | $5.718\times10^{-7}$ |
| 0.6 | 2.584 | 2.584000562 | $5.623\times10^{-7}$ |
| 0.7 | 3.107 | 3.107000401 | $4.009\times10^{-7}$ |
| 0.8 | 3.688 | 3.688000293 | $2.929\times10^{-7}$ |
| 0.9 | 4.321 | 4.321000289 | $2.887\times10^{-7}$ |
| 1.0 | 5.000 | 5.000000200 | $1.999\times10^{-7}$ |

**Table 3.** Numerical Results for Example 3

| X | Exact solution | Proposed method | Error |
|---|---|---|---|
| 0 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 1.105170918 | 1.105255338 | $8.442\times10^{-5}$ |
| 0.2 | 1.221402758 | 1.221484948 | $8.219\times10^{-5}$ |
| 0.3 | 1.349858808 | 1.349918828 | $6.002\times10^{-5}$ |
| 0.4 | 1.491824698 | 1.491867908 | $4.321\times10^{-5}$ |
| 0.5 | 1.648721271 | 1.648764281 | $4.301\times10^{-5}$ |
| 0.6 | 1.822118800 | 1.822155230 | $3.663\times10^{-5}$ |
| 0.7 | 2.013752707 | 2.013787117 | $3.441\times10^{-5}$ |
| 0.8 | 2.225540928 | 2.225571418 | $3.049\times10^{-5}$ |
| 0.9 | 2.459603111 | 2.459629491 | $2.638\times10^{-5}$ |
| 1.0 | 2.718281828 | 2.718281934 | $1.055\times10^{-7}$ |

$x_k = a + \frac{(b-a)}{N}k, \quad \mathrm{k} = 0,1,\ldots,\mathrm{N}$ to obtain

$$M_j + \sum_{j=1}^{N}\int_0^1 W(x_k, t_j + qh)\, F\Big\{u_j + \frac{M_j}{\tau} + \Big[\frac{1}{h}(u_{j+1} - u_j) + \frac{1}{\tau\theta}(M_{j+1} - M_j)\Big]qh + \frac{1}{\tau^2 \sin\theta}(M_j\cos\theta - M_{j+1})\sin\tau(qh)$$
$$- \frac{M_j}{\tau^2}\cos\tau(qh)\Big\}dq = h(x_k), \quad \mathrm{j} = 0,1,\ldots,\mathrm{N}+1 \qquad (26)$$

Similarly, (21) becomes

$$M_{j-1} + \sum_{j=1}^{N}\int_0^1 W(x_k, t_{j-1} + qh)\, F\Big\{u_{j-1} + \frac{M_{j-1}}{\tau^2} + \Big[\frac{1}{h}(u_j - u_{j-1}) + \frac{1}{\tau\theta}(M_j - M_{j-1})\Big]qh + \frac{1}{\tau^2 \sin\theta}(M_{j-1}\cos\theta - M_j)\sin\tau(qh)$$
$$- \frac{M_{j-1}}{\tau^2}\cos\tau(qh)\Big\}dq = h(x_k), \quad \mathrm{j} = 1,2,\ldots,\mathrm{N} \qquad (27)$$

Equations 26 and 27 together with Equation 3 form set of (N+ 2) algebraic linear equation in (N+2) unknowns.

## NUMERICAL EXAMPLES

We consider here the following examples on linear and non linear Fredholm - integro differential equations. These examples have been chosen from Abdul-Majid (2011).

### Example 1

$$u''(x) = \frac{5}{3} - 11x + \int_0^1 u(t)\,dt$$
$$u(0) = u'(0) = 1$$
$$u(x) = 1 + x + \frac{5}{6}x^2 - \frac{5}{3}x^3$$

(Table 1)

### Example 2

$$u''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} x\,t\,u^2(t)\,dt$$
$$u(0) = 1, \quad u'(0) = 0.$$
$$u(x) = 1 + 5x^2 - x^3.$$

(Table 2)

### Example 3

$$u''(x) = e^x + \frac{1}{4}(e^2 - 2)x + \frac{1}{2}\int_0^1 x\left(t - u^2(t)\right)dt$$
$$u(0) = u'(0) = 1$$
$$u(x) = e^x$$

(Table 3)

### Example 4

$$u''(x) = \frac{1}{2}\int_{-1}^{1}(1 - x^2 t)(u(t) - u^2(t))dt$$
$$u(0) = u'(0) = 0$$
$$u(x) = \frac{5(\lambda - 6)}{3\lambda}x^2.$$

**Table 4.** Numerical Results for Example 4.

| X | Exact solution $\frac{(\lambda - 6)}{\lambda}$ | Exact solution $\lambda = 10$ | Proposed method $\lambda = 10$ | Error |
|---|---|---|---|---|
| 0 | 0.000000000 | 0.000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.016666667 | 0.006666667 | 0.0066676308 | $9.638\times10^{-7}$ |
| 0.2 | 0.066666667 | 0.026666667 | 0.0266750090 | $8.342\times10^{-6}$ |
| 0.3 | 0.150000000 | 0.060000000 | 0.0600068930 | $6.893\times10^{-6}$ |
| 0.4 | 0.266666667 | 0.106666667 | 0.1066713340 | $4.667\times10^{-6}$ |
| 0.5 | 0.416666667 | 0.166666667 | 0.1666967570 | $3.009\times10^{-5}$ |
| 0.6 | 0.600000000 | 0.240000000 | 0.2400299400 | $2.994\times10^{-5}$ |
| 0.7 | 0.816666667 | 0.326666667 | 0.3266945570 | $2.789\times10^{-5}$ |
| 0.8 | 1.066666667 | 0.426666667 | 0.4266884770 | $2.181\times10^{-5}$ |
| 0.9 | 1.350000000 | 0.540000000 | 0.5400166800 | $1.668\times10^{-5}$ |
| 1.0 | 1.666666667 | 0.666666667 | 0.6666667679 | $1.009\times10^{-7}$ |

(Table 4)

## CONCLUSION

We have shown that approximate solutions of Fredholm - integro differential equations by using spline collocation method are obtained. The results obtained are compared with the exact results (Tables 1 to 4). Spline collocation method therefore, is a powerful procedure for solving both linear and non linear Fredholm – integro differential equations. The results obtained are relatively close to the exact solution. This confirmed the reliability and effectiveness of the method to handle problems within this class.

### REFERENCES


**Abdul-Majid W (2011).** Linear and Non linear Integral Equations: Methods and Applications. Springer Heidelberg Dordrecht, London. New York.

**Ahlberg JN, Nilson EN, Walsh JL (1976).** The theory of spline and their application. Academic press inc. Springer.

**Awadeh F, Adawi A, Al-Shara S (2009).** Numerical methods for solving non-linear integral equations. International Forum 4(17):805-817.

**Brumer H, Tang T (1989).** Polynomial Spline Collocation methods for the non linear Basset equations. J. Comput. Math. Appl. 18(5):449-457.

**Craig AT, Harold W (1994).** Fredholm determinants, differential equations and Matrix Models.Commun. Maths. Phys. 163:33-72.

**Davis HT (1962).** Introduction to non linear Differential and Integral equations. Dover, New York.

**Gegele OA (2004).** Numerical Solution of Second Order Partial Differential Equations by cubic Spline Tau Method. (Master's Dissertation) Unpublished, University of Ilorin, Nigeria

**Golberg MA (1973).** Convergence of an Initial value Methods for Fredholm Integral equation. J. optimiz. theor. Appl. 4(12):344-354.

**Hesan-Eldeen OG, Hossein M (2012).** Numerical Solution of Functional integral and integro – differential Equations by using B – Spline. Journal of Scientific Research (http://www.scirp.org/journal/am).

**Joseph AG, Gene Ak (1973).** New Error Analysis for a Cubic Spline Approximate Solution of a Class of Volterra – Integro Differential Equations. J. Math. Computat. 27:568:570.

**Kanwal RP (1997).** Linear Integral Equations. Birkhauser, Boston.

**Karkarashvili GS (1993).** Fredholm Integro – Differential Equation. Mathematics Institute, Georgian Acad. Sci. 64:6-16.

**Laidlaw BH, Phillips GM (1972).** Iterative Methods for Solving Fredholm integral Equations. BIT, s Problems and Exercises, Kluwer, Boston. 12:188-193.

**Linz P (1985).** Analytical and Numerical Methods for Volterra Equations, SIAM, Philadelphia

**Micula G, Pavel P (1992).** Differential and Integral equations Through Practical Problems and Exercises. Kluwer, Boston.

**Ogunlaran OM (2012).** Spline Approximation methods for solving some ordinary and partial differential Equations (Doctoral thesis unpublished), University of Ilorin, Nigeria.

**Ogunlaran OM, Taiwo OA (2013).** Spline Methods for a class of Singularly Perturbed Boundary value Problems. Int J. Math. Res. 2(1):1-10.

**Sastry SS (2000).** Introductory methods of Numerical Analysis. 3rd Edition, Prentice Hall. New Delhi.

**Sastry SS (1976).** Finite Difference Approximation to one Dimensional Parabolic Equation using cubic Spline Technique. J. Computat. Appl. Math. 2(1):23-26.

**Vilmos H (1999).** On Polynomial Spline Collocation Methods for Neutral Volterra I tegro – Differential Equations with Delay Arguments. Proceeding of the 1st Conference on Applied Mathematics and computation. Dubrovnik, Croatia.