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Research Paper

Learners' common errors in implicit differentiation: An analytical study

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Abstract. The purpose of this study is to analyze learners' errors in implicit differentiation and identify common errors committed by 20% or more of Common First Year learners (117 Male) at King Saud University. The researcher prepared a test in implicit differentiation consisting of seven various questions covering almost all ideas of implicit differentiation contained in the textbook (Differential Calculus Math101). By analyzing the learners' answers, the researcher concluded two types of learners' common errors, namely Algebra Errors (AE) and Calculus Errors (CE). Algebra Errors appeared when learners isolated the common factor, collected similar terms, simplified Algebra fractions, multiplied Algebra terms, dealt with exponentials and roots functions, and found y', and Calculus Errors that appeared when learners applied the Chain rule, applied a multiplication rule, found the derivative of any constant, and differentiated functions.

Keywords: Implicit differentiation, functions, common errors.

INTRODUCTION AND THEORETICAL FRAMEWORK

Mathematics is the queen of sciences but unfortunately, learners fear it in spite of the fact that mathematics is very essential to the growth of many other disciplines. The science of mathematics depends on mental ability. It is the means to develop thinking and intelligence, and sharp the mind and make it creative. The development of human beings and their culture depends on the development of mathematics. That is why it is known as the base of human civilization. It is also the language of all sciences and the center of all engineering branches which revolve around it. Therefore, it is the past, present and future of all sciences (Yadav, 2017). Mathematics is important for life and supports all-round personal development. Mathematics significantly influences pupils' and learners' education both in a special branch (mathematical knowledge) and in terms of moral education. We can find mathematical applications in nature, technology, architecture, machinery, building industry, banking sector, research, and cartography. Very interesting applications in genetics that utilize mathematics in addition to its applications in nature. Statistical methods are used in hypothesis testing in genetics (Hodaňová and Nocar, 2016).

Mathematics is divided into two branches: Pure Mathematics, which is concerned with increasing knowledge of the subject rather than using knowledge in practical ways, for example, trigonometry, geometry, set theory, vector, etc.; and Applied Mathematics, which is concerned with using knowledge of pure mathematics in practical ways, for example, mechanics, dynamics, statics, and physics, etc.

Calculus is the study of differentiation and integration (this is indicated by the Chinese translation of "Calculus"). Both concepts of differentiation and integration are based on the idea of limit. Calculus was "invented" by Newton and Leibniz independently in the late 17th century. Newton used ˙ whereas Leibniz used $\frac{dy}{dt}$ to denote the derivative of x with respect to t (time). The notation y' is simple whereas $\frac{dy}{dx}$ reminds us that it is defined as a limit of difference quotient (Chung, 2014).

Calculus is a branch of mathematics taught Limits and differentiation, integration and infinite series, and it is the science used to study the change in the functions and analysis. It has two major branches, differential Calculus (concerning rates of change and slopes of curves), and integral Calculus (concerning accumulation of quantities and the areas under and between curves). These two branches are related to each other by the fundamental theorem of Calculus. Both branches make use of the fundamental notions of convergence of infinite sequences and infinite series to a well-defined limit. Today, Calculus has widespread uses in science, engineering and economics.

Research has shown that learners have difficulties with many topics in Calculus. Such difficulties include, but are not limited to, recognizing functions, knowing the derivative rules and knowing when they need to be applied, and a lack of understanding of what derivatives are. It is also possible that other issues in relation to the derivative may come up when working on implicit differentiation problems because the usage of implicit differentiation often requires learners to use the chain rule. It is possible that the areas of difficulty learners have with the chain rule may resurface when they solve implicit differentiation problems (Chu, 2019).

Implicit differentiation is important as it allows us to take the derivative of an equation with multiple variables without having to worry about whether the equation is a function, and it is a technique that allows us to differentiate equations that are not explicit functions.

An explicit function is a function in which one variable is defined only in terms of the other variable. Explicit functions are functions where we know exactly what y is in terms of x. Here are some examples of explicit functions:

a) $y = x^2 + 3$ Parabola

b) $y = \sqrt{4 - x^2}$ Top half of the circle centered at the origin with radius 2.

c) $y = -\sqrt{4 - x^2}$ Bottom half of the circle centered at the origin with radius 2.

An implicit function is a function in which one variable is not defined only in terms of another variable. Implicit functions do not tell us what y is in terms of x. Here are some examples of implicit functions.

a) $y - x^2 = 3$ Parabola

b) $y^2 + x^2 = 4$ Circle centered at the origin with radius 2.

c) $x^3 + y^3 = 6xy$

Some implicit functions can be written explicitly: both examples (a) represent the same parabola and (b) can be solved for either the top or the bottom of the circle. (c) cannot be solved explicitly for y in terms of x. We want to be able to differentiate functions that either cannot be written explicitly in terms of x or the resulting function is too complicated to deal with. To do this, we use implicit differentiation.

In mathematics education, it is often helpful to establish the breadth and depth of learners' understanding of mathematical concepts through error analysis and diagnosis (EAD), to commence teaching at a learner's conceptual level. EAD aims to analyze, expose and interpret error patterns to modify instruction in reaction to them (Nyaumwe, 2008).

The central problem of ineffective mathematics instruction is exacerbated firstly by teacher's unawareness and unfamiliarity with general or specific learners' errors and misconceptions on key mathematics concepts and competencies. Analyzing errors made by learners provides teachers with insight regarding their learners' procedural and conceptual misunderstandings (Mercer and Mercer, 2005). The errors that learners make can sometimes be even more informative to teachers than correct responses. Learners' errors often provide insights into learners' misunderstandings about a particular mathematics concept or skill. Dali (2008) has advised that it is not wise for teachers and parents to openly criticize a learner's error. He argued that it is wiser to first understand the learners' errors and then find the wise ways to sublimate them so that the learner could correct them for him/herself. Thus, errors and misconceptions are viewed as opportunities for deepening **one's understanding of** learners' difficulties and are of critical importance in furthering learning.

Research studies revealed that many learners have not developed clear concepts in Mathematics and that some of them use Calculus notation informally. A study by Siyepu (2015) showed that errors displayed by learners were conceptual and procedural; there were also errors of interpretation and linear extrapolation. The findings revealed that the participants were not familiar with basic operational signs such as addition, subtraction, multiplication and division of trigonometric functions. The participants demonstrated a poor ability to simplify once they had completed differentiation. Bicer *et al*. (2014) indicated that several pre-service teachers struggled with representing inequalities solutions in the number line. They added or excluded values in their solutions by drawing a blurred circle on a number line instead of an empty circle. Learners also made basic arithmetic errors. The most common errors were addition, subtraction, multiplication, division and the distributive property. Muzangwa and Chifamba (2012) analyzed errors and misconceptions in an undergraduate course in Calculus. The analysis of the results from the tests showed that the majority of the errors were due to knowledge gaps in basic Algebra. The researchers noted that errors and misconceptions in Calculus were related to learners' lack of advanced mathematical thinking since concepts in Calculus are intertwined; they also highlighted some common errors which can be committed by lecturers during the teaching process. The results of a study by Ciltas and Tatar (2011) indicated that learners have difficulties in forming a correct solution set and could not fully understand the concept of absolute value, and learners experienced difficulties in applying the basic arithmetic operations and interpreting the interval that is founded correctly in inequality questions. According to Alshare' and Alabed (2010), they revealed some common errors, such as misconceptions, confusing an inequality with an equation, using commutative multiplication in solving inequalities, and changing the direction of inequality when multiplying by a negative number. Some other calculation errors and careless errors were also recorded. The common errors ranged between 5.7% for changing the direction of inequality when multiplying by a negative number, and 22.5% for conceptual errors. The study recommended that faculty members emphasize the subject of inequalities for freshmen and administer tests in order to categorize them and develop the appropriate treatment plans. Orhun (2010) states that learners registered for Calculus in their first year at a university perform badly in the operations of trigonometric expressions; for example, learners demonstrate difficulties in the multiplication of $\sin x . \sin x$, and he argues that this may be because there may not be much emphasis on the learning of addition, subtraction, multiplication and division of trigonometric functions in the secondary school curriculum. A study by Alqudah (2008) aimed to classify the common errors of the second secondary learners' answers in mathematics 2006-2007 in Jourdan. The errors were classified into some categories by two stages; the first in each question of the exam and the second in all questions of the exam. The most common errors were classified into four main errors; conceptual errors, errors related to laws and theories, errors related to calculations and logical operations, and errors related to the wrong hypothesis.

The results of the study of Ureyen *et al.* (2006) showed that learners are unable to successfully resolve the inequalities and that many of them multiply the disparate ends of a certain amount of "cross-multiplication" without interest in a sign of a variable. The study recommended providing feedback to learners to know their errors and take advantage of them. Engelbrecht *et al.* (2005) have argued that over-reliance on procedural knowledge in solving Calculus problems results in many errors and misconceptions and ultimately failure in this subject. This is because procedural knowledge addresses how to do a task only. It cannot help to answer the question of what is to be done and why, which is often required in most mathematics questions.

Melis (2003) studied how learners learn from wrongly worked mathematical examples to enhance understanding of problematic mathematics concepts, and Monson and Judd (2001) designed a diagnostic system that assessed learners' misconceptions in Calculus. While Bezuidenhout (2001) postulated that the oversimplified Calculus exercises found in many textbooks often encouraged procedural thinking at the expense of the conceptual understanding of Calculus, he argued that this played a crucial role in learners‟ difficulties with the Calculus.

As for learners' errors in implicit differentiation, in particular, Mirin and Zazkis (2019) presented a conceptual basis for differentiating an equation, to make implicit differentiation more explicitly defined. They explain that when performing implicit differentiation, it is important to clearly define any implicit functions and then recognize that the two functions set equal in an equation may only be equal on a restricted domain. They explain that only once it is clear that the two functions are equal to each other can one understand why their derivatives, with respect to the chosen independent variable, are also equal to each other, Chu (2019) suggested that implicit differentiation is challenging for learners. Approximately 50% of the survey responses to implicit differentiation problems were correct. The interviews suggested that some learners had a strong understanding of implicit differentiation and others did not. Learners who have a strong understanding of the idea of implicit differentiation appear to be more successful in carrying out the procedures. Jones (2017) found that learners - in his study - often thought that implicit differentiation must be required for all applied derivatives. Also, Dawkins and Epperson (2014) concluded that the learners who lacked proficiency in reasoning in the Algebraic and graphical registers would be underprepared for a Calculus course; in their research, most learners including the top-performing learners heavily depended on the Algebraic method; and learners who excelled in the course were the ones who had the good study habit of learning both conceptually and procedurally.

In addition, a study by Kakoma and Makonye (2010) showed that learners had problems in understanding the meaning of the derivative when it appeared as a fraction or the sum of two parts. Learners also showed difficulties with applying the chain rule for differentiation and use of the parameters in partial differentiation. Infante (2007) found that it is especially difficult for learners to recognize the implicit variable of time within related rates problems. She found that using a dynamic computer program, which illustrated the related rates problems the learners were working through, helped learners to recognize the implicit variable of time and apply the chain rule. She also found that if learners used the chain rule in a related rates problem, it became a way for them to create a delta equation, which helped them to better see the relationship between the given and unknown rates in the problem. Visualizing this relationship in the equation ultimately helped them to successfully solve the related rates problem. In other words, she found that the chain rule was a main factor for solving a related rates problem. Similarly, the survey results of Martin (2000) showed that learners struggle with implicit differentiation problems. It appears that they have more difficulty with some types of these problems than others (such as the ones involving product rule and chain rule) and it appears that most errors were Calculus-based, but it was ambiguous in some cases as to where these difficulties stem from when examining written work alone.

Speer and Kung (2016) demonstrated that research on implicit differentiation is missing from mathematics education research, and they emphasized that there is a need for research on implicit differentiation to supplement the existing literature on the derivative and chain rule.

So, the researcher has been intrigued by how learners' errors in mathematics affect their mathematics performance in general and in certain tasks (implicit differentiation) in particular. Learners' errors constitute challenges to reaching curriculum outcomes. But, they can also be viewed in a positive light. It is crucial that if there should be effective teaching of mathematics, teachers understand the nature of learner mathematical errors so that they can design appropriate strategies to help learners understand mathematics. So, before Calculus learning should be improved, the nature of the problem of learning Calculus should first be understood through studying the errors that learners show in examination scripts. These must, first of all, be fully understood and their characteristics well documented before the teaching of Calculus can be effective. This assessment is best done by analyzing the errors that learners show in their artefacts such as written answers to mathematics questions. These led the researcher to study and diagnose the common errors, and classify them among learners of the preparatory year at King Saud University who study Calculus.

Questions

- What are the learners' common errors in implicit differentiation?
- What are the learners' common errors in each question of implicit differentiation test?

Objectives

This study aims to:

1. Identify common learners' errors in implicit derivation in general.

a. Determining the type of learners' common errors in each question of the implicit differentiation test that prepared for this study in detail.

METHOD

Mathematics education researchers need to be responsive to classroom challenges faced by teachers and learners through undertaking grounded research that produces scientific-based evidence helping to determine concrete decisions to improve learning and performance. In this regard, this study hoped to add expertise to mathematics education professional practice through investigating, examining, exposing and evaluating common errors that come out of learners in implicit differentiation. This study posits that research that problematizes learner errors is learner-centered and holds promise for upgrading teaching and learning of Calculus in general and implicit differentiation in particular. One product of this research is a register and database of errors in implicit differentiation. The results of this study will be available to education policymakers and educationists. It would be beneficial to explore how learners think about implicit differentiation and the areas where they struggle when working with it. This would help us understand how learners' thinking on related topics, such as differentiation and the chain rule, could be used to explain difficulties they have with implicit differentiation as well as if any unique difficulties stem from implicit differentiation itself. Knowing these things could help to identify where learners' understanding of this topic, and related ones, could be improved.

Procedural definitions

Learners: all-male learners who have been accepted in the scientific track in the preparatory year program at King Saud University, totaling (1198) learners who completed the study of Calculus Course (Math101) in the first semester (2018-2019).

Learners' common error: an error that the proportion of its prevalence among learners is more than (20%).

Research methodology

Descriptive approach has been used to get the data and facts about the nature of the learners' common errors and ratios.

Research sample

The study sample consisted of (117) male learners, distributed into (5) sections; it has been selected randomly from male learners (1198) in the Common First Year at King Saud University in the first semester (2018-2019).

Tools

The study tool is the implicit differentiation test, the

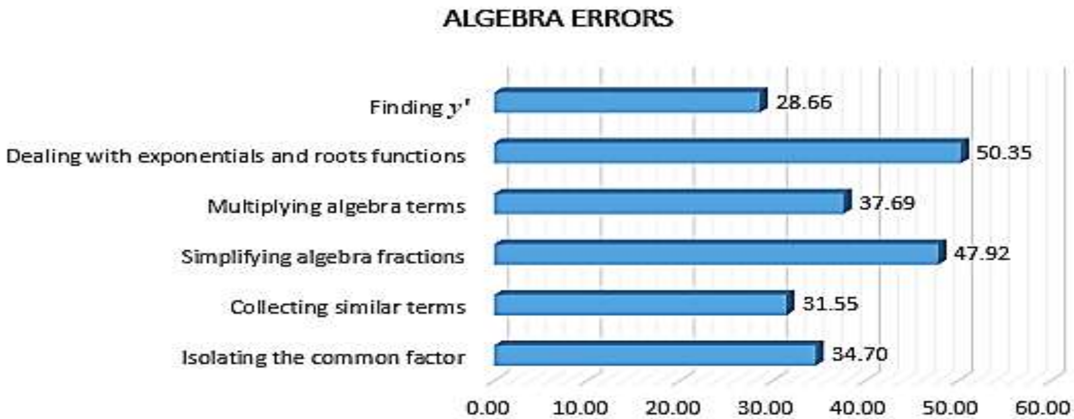

Figure 1. Algebra errors.

researcher built this test in light of the expected errors and types of implicit differentiation functions. The test included (7) essay items (open-ended questions). To check the validity of the test, it was presented to a group of arbitrators, four PhDs in curriculum and methods of teaching mathematics, two PhDs in educational measurement and evaluation and (8) instructors of mathematics who have master's degrees in mathematics. After reviewing the opinions of the arbitrators and suggestions, modifications have been made. Reliability was computed by using a test and re-test by applying the test on an exploratory sample consisting of (35) learners who completed studying the implicit differentiation. The Pearson correlation coefficient was (0.83) between the average performance of learners in the first time and when repeating it.

Statistical analysis

The frequencies and percentages were extracted to answer the questions of this study.

RESULTS

This study sought to determine the types of errors that learners made when solving implicit differentiation problems, so the researcher analyzed learners' errors in an implicit differentiation test that was prepared for this study. Accordingly, the researcher found two types of errors that learners committed generally.

The first type is Algebra errors (AE) which includes: isolating the common factor, collecting similar terms, simplifying Algebra fractions, multiplying Algebra terms, dealing with exponentials and roots functions, and finding y'. The following figure shows the averages of percentages of common Algebra errors that learners make while solving implicit differentiation questions.

It is clear from Figure 1 that the highest percentage of learners' errors was dealing with exponentials and roots functions and converting each to the other, followed by simplifying Algebra fractions, then multiplying the Algebra terms, and the lowest error percentage was in finding the derivative in its final form.

The second type is Calculus Errors (CE) which includes: applying the Chain rule, applying a multiplication rule, finding the derivative of any constant, and differentiating functions. The following figure shows the averages of percentages of common calculus errors that learners made while solving implicit differentiation questions.

Figure 2 shows that the highest error percentage among learners was applying the chain rule, followed by differentiating fractions, then applying a multiplication rule, as the lowest error percentage was in differentiating a constant.

To answer the second question, the researcher examined the learners' answers to the questions of the implicit differentiation test. He reached a set of Algebra and Calculus errors, as follow: (note: percentages of the mentioned number of learners are calculated based on the number of learners who reached the mentioned step, not from the total number of learners)

Question 1:

$$x^2y + xy^2 = 3x$$

$$Solution: y' = \frac{dy}{dx}$$

$$(2xy + x^2y') + (y^2 + 2xyy') = 3 \text{....Take the derivative of individual term}$$

$$x^2y' + 2xyy' = 3 - 2xy - y^2 \text{............Get all of the terms with } y' \text{ onto one side of the equation}$$

$$y'(x^2 + 2xy) = 3 - 2xy - y^2 \text{...........Factor } y' \text{ out on the left side of the equation}$$

$$y' = \frac{3 - 2xy - y^2}{x^2 + 2xy} \text{...............................Isolate } y' \text{ giving you the solution}$$

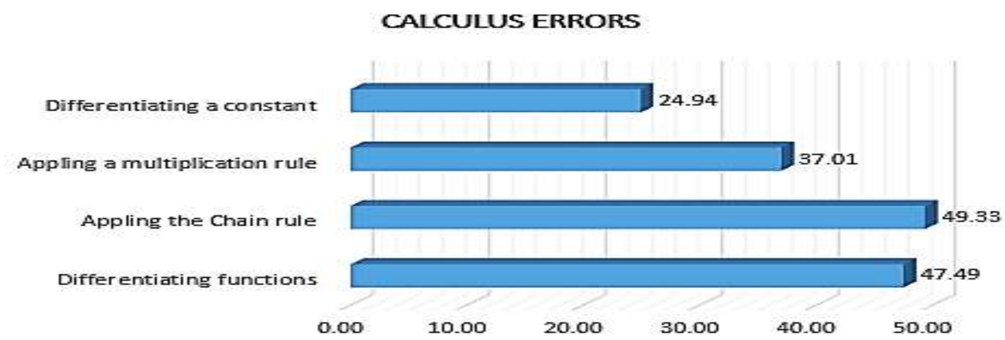

Figure 2. Calculus errors.

It is clear through the learners' solutions to the first question that 21 learners out of 76 (27.63%) made the first algebra error when collecting similar algebra terms in step2, though they wrongly wrote terms as $x^2y'-2xyy'=3-2xy-y^2$ or $x^2y'+2xyy'=3+2xy+y^2$. The second algebra error is isolating the common factor wrongly, as 17 learners out of 47 (36.17%) could not isolate the common factor in step3. The third algebra error is finding y', though 8 learners out of 29 (27.59%) wrongly found y' as $y'=\dfrac{x^2+2xy}{3-2xy-y^2}$ or $y'=\dfrac{3-y^2}{x^2}$.

It is also evident through the learners' solutions to this question that 67 learners out of 117 (57.26%) made the first calculus error in step 1 when differentiating implicit functions. They wrongly differentiated $x^2y+xy^2=3x$ as $2x+2y=3$ in a direct way; however, they correctly differentiated x^2, x and $3x$ as $2x, 1$ and 3. The second calculus error is that they did not apply the chain rule in step1, though 67 learners out of 117 (57.26%) wrongly differentiated y and y^2 as 1 and $2y$. The third calculus error is that they did not apply the multiplication rule in step1, as 24 learners out of 50 (48%) differentiated x^2y and xy^2 as $2xy'$ and $2yy'$.

Question 2:

$$(2x+y)^2=y$$
$$Solution:$$
$$2(2x+y)(2+y')=y' \text{..................Take the derivative of individual term}$$
$$(4x+2y)(2+y')=y'$$
$$8x+4xy'+4y+2yy'=y' \text{.........Simplify the left side}$$
$$4xy'+2yy'-y'=-8x-4y \text{.........Get all of the terms with } y' \text{ onto one side}$$
$$y'(4x+2y-1)=-8x-4y \text{..........Factor } y' \text{ out on the left side}$$
$$y'=\frac{-8x-4y}{4x+2y-1} \text{..............................Isolate } y' \text{ giving you the solution}$$

By examining the learners' solutions to question 2, it became clear that the first algebra error is multiplying algebra terms, though 26 learners out of 67 (38.81%) wrote step 2 as $4x+y$. The learners only multiplied 2 by the first term, and they wrote step 3 as $8x+2yy'$; they multiplied the first term by the first and the second term by the second. The second algebra error is collecting similar algebra terms, as 18 learners out of 56 (32.14%) wrongly collected the similar terms in step 4; they moved the terms to the other side of the equation with the wrong signs, for example, $8x+4y$ and y'. The third algebra error is isolating the common factor y', though 16 learners out of 44 (36.36%) wrongly isolated the common factor y'. For example, they wrote this step as $y'(4x+2y)=-8x-4x$. The fourth algebra error is finding y', though 11 learners out of 31 (35.48%) wrongly found y' as $y'=\dfrac{4x+2y-1}{-8x-4y}$ or $y'=\dfrac{-2-2}{-1}=4$.

It is also evident through the learners' solutions that 48 learners out of 97 (49.48%) made the first calculus error in step1 when differentiating implicit functions. For example, they wrongly differentiated $(2x+y)^2$ as $(2+1)^2=9$, and y as 1. The learners did not apply the chain rule or apply it incorrectly; for example, they wrongly differentiated $(2x+y)^2$ as $2(2x+y)$ or $2(2x+y)+(2+y')$, and y as yy'.

Question 3:

$$(x^2+1)^2=(3y)^2$$
$$Solution:$$
$$(x^2+1)^2=9y^2 \text{.............Simplify the right side}$$
$$2(x^2+1)(2x)=18yy' \text{...Take the derivative of individual term}$$
$$4x^3+4x=18yy' \text{.........Simplify the left side}$$
$$y'=\frac{4x^3+4x}{18y} \text{..................Isolate } y' \text{ giving you the solution}$$

Through the learners' solutions to question 3, it became clear that 46 learners out of 102 (45.10%) made the first algebra error in step1 when applying the exponent rules, though these learners wrote $(3y)^2$ as $3y^2$ or $9y$, $6y^2$. The second algebra error appeared when learners multiplied $2(x^2+1)(2x)$, though 37 learners out of 102 (36.27%) wrote this step as x^2+1+4x or gave another wrong answer. The third algebra error appeared when learners collected the same terms, though 18 learners out of 57 (31.58%) wrote $4x^3+4x=18yy'$ as $8x^4=18yy'$ or $-18yy'=4x^3+4x$. The fourth algebra error appeared when learners found y', though 13 learners out of 39 (33.33%) wrongly found y' as $y'=\dfrac{18y}{4x^3+4x}$ or $y'=\dfrac{8x^4}{18y}$, $y'=-\dfrac{4x^3+4x}{18y}$.

The first calculus error appeared when learners differentiated implicit functions, though 62 learners out of 98 (63.27%) wrongly differentiated $(x^2+1)^2$ as $(2x)^2$ or $2(x^2+1)$, $2(x^2+1)(2x+1)$, and $9y^2$ as $18y$ or $18y'$. These learners did not apply the chain rule or wrongly applied it. The second calculus error appeared when learners differentiated the constant (1), though 27 learners out of 98 (27.55%) differentiated x^2+1 as $2x+1$.

Question 4:

$$\frac{xy}{x^2+y^2}=x+1$$

Solution:

$(x^2+y^2)(x+1)=xy$...............................Multiply both sides by x^2+y^2

$x^3+x^2+xy^2+y^2=xy$...........................Simplify the left side

$3x^2+2x+(y^2+2xyy')+2yy'=y+xy'$..Take the derivative of individual term

$2xyy'+2yy'-xy'=y-3x^2-2x-y^2$........Get all of the terms with y' onto one side

$y'(2xy+2y-x)=y-3x^2-2x-y^2$..........Factor y' out on the left side

$y'=\dfrac{y-3x^2-2x-y^2}{2xy+2y-x}$.....................................Isolate y' giving you the solution

By examining the learners' solutions to question 4, it was found that 46 learners out of 96 (47.92%) did not multiply both sides of the equation by x^2+y^2 to simplify it. The second algebra error appeared when the learners multiplied terms in step3, though 19 learners out of 50 (38%) wrote this step as $x^3+y^2=xy$. The third algebra error appeared when learners collected similar terms (step 4), though 17 learners out of 50 (34%) wrongly collected the similar terms as they moved some of terms to the other side of the equation with the same signs (for example $2x, y^2$ and xy'). The fourth algebra error is isolating the common factor, though 12 learners out of 38 (31.58%) could not isolate the common factor (y'). The fifth algebra error appeared when learners found y', though 9 learners out of 27 (33.33%) wrongly found y' as

$$y'=\frac{2xy+2y-x}{y-3x^2-2x-y^2}$$ or $$y'=\frac{y-3x^2-2x-y^2}{2xy+2y-x}=\frac{-3x-y}{2}.$$

It is also clear that 46 learners out of 96 (47.92%) made the first calculus error in step 1, they differentiated the functions in a direct way, though they wrongly differentiated $\dfrac{xy}{x^2+y^2}=x+1$ as $\dfrac{(1)(1)}{2x+2y}=1$ or $\dfrac{(x^2+y^2)+xy(2x+2y)}{(x^2+y^2)^2}=1$ by the quotient rule. The second calculus error appeared when learners differentiated implicit functions, though 48 learners 96 (50%) wrote the derivative of xy as $(1)(1)$ or $x+y$ and y^2 as $2y$. They did not apply the chain rule or apply it wrongly. The third calculus error is that they did not apply the multiplication rule, though 21 learners 96 (21.88%) wrongly differentiated xy as $(1)(1)$ or $(1).y'=y'$.

Question 5:

$$\sqrt{x}+\sqrt{y}=9$$

Solution:

$x^{1/2}+y^{1/2}=9$..............................Convert root functions to exponential functions

$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}y'=0$..............Take the derivative of individual term

$\frac{1}{2}y^{-1/2}y'=-\frac{1}{2}x^{-1/2}$..................Get all of the terms with y' onto one side

$y^{-1/2}y'=-x^{-1/2}$....................Multiply both sides by 2

$y'=\dfrac{-x^{-1/2}}{y^{-1/2}}$..................................Isolate y' giving you the solution

$y'=-\dfrac{y^{1/2}}{x^{1/2}}=-\dfrac{\sqrt{y}}{\sqrt{x}}=-\sqrt{\dfrac{y}{x}}$..Simplify the right side

It appears from the solutions of learners that 63 learners out of 103 (62.38%) did not convert root functions to exponential functions (1st Algebra error). The second algebra error appeared when learners collected similar terms (step3), though 27 learners out of 81 (33.33%)

moved terms to the other side of the equation with the same signs, for example, $\frac{1}{2}x^{-1/2}$ or $\frac{1}{2}x^{1/2}$. The third algebra error appeared when learners found y', though 7 learners out of 32 (21.88%) wrongly found y' as $y' = \frac{y^{-1/2}}{-x^{-1/2}}$. The fourth algebra error appeared when learners were unable to perform step 6, though 6 learners out of 25 (24 %) did not know how to convert the exponential functions to the root functions, and the properties of roots.

It is also clear that 56 learners out of 103 (54.37%) made the first calculus error in step2. They differentiated a function in a direct way, though they wrongly differentiated $x^{1/2}$ as $\frac{1}{2}x^{1/2}$ or $\sqrt{x}$ as $\frac{1}{2\sqrt{x}}$. The second calculus error appeared when learners differentiated implicit functions. Thirty-four (34) learners out of 84 (40.48%) wrote the derivative of $y^{1/2}$ as $\frac{1}{2}y^{-1/2}$ or $\frac{1}{2}y^{1/2}$ or $\frac{1}{2}(y')^{-\frac{1}{2}}$, and $\sqrt{y}$ as $\frac{1}{2\sqrt{y}}$ or $\frac{1}{2\sqrt{y'}}$, though they did not apply the chain rule. The third calculus error appeared when learners differentiated the constant (9), though 23 learners out of 103 (22.33%) did not differentiate the constant (9).

Question 6:

$x\sin y + y\sin x = x$

Solution:

$\sin y + x(\cos y)y' + (\sin x)y' + (\cos x)y = 1$.........Take the derivative of individual term

$xy'\cos y + y'\sin x = 1 - \sin y - y\cos x$..................Get all of the terms with y' onto one side

$y'(x\cos y + \sin x) = 1 - \sin y - y\cos x$..................Factor y' out on the left side

$y' = \frac{1 - \sin y - y\cos x}{x\cos y + \sin x}$..Isolate y' giving you the solution

Through the learners' solutions for question 6, it became clear that 26 learners out of 85 (30.59%) wrongly collected similar terms (step 2) as they moved $\sin y$ and $y\cos x$ to the other side of the equation with the same signs ($\sin y + y\cos x$). The second algebra error appeared when learners found y', though 7 learners out of 33 (21.21%) wrongly found y' as $y' = \frac{x\cos y + \sin x}{1 - \sin y - y\cos x}$.

It is also clear that 27 learners out of 85 (31.76%) made the first calculus error in step 1, they differentiated the function in a direct way, though they wrongly differentiated $x\sin y + y\sin x$ as $(1)\cos y + (1)\cos x = \cos y + \cos x$. The second calculus error appeared when learners differentiated implicit functions, 23 learners out of 55 (41.81%) wrote the derivatives of $\sin y$ and y as $\cos y$ or $\cos y'$ and y', though they didn't apply the chain rule or wrongly applied it. The third calculus error is didn't apply the multiplication rule, though 32 learners out of 85 (37.65%) wrongly differentiated $x\sin y + y\sin x$, for example, the derivative of $x\sin y + y\sin x$ is $(1)(y'\cos y) + (y')(\cos x) = y'\cos y + y'\cos x$.

Question 7:

$\sqrt{1+\cos^2 y} = xy$

Solution:

$(1+(\cos y)^2)^{1/2} = xy$..Convert the root function to exponential function

$\frac{1}{2}(1+(\cos y)^2)^{-1/2}(2\cos y)(-\sin y)(y') = xy' + y$...Take the derivative of both sides

$-\sin y\cos y(1+(\cos y)^2)^{-1/2}y' - xy' = y$....................Get all of the terms with y' onto one side

$y'(-\sin y\cos y(1+(\cos y)^2)^{-1/2} - x) = y$..................Factor y' out on the left side of the equation

$y' = \frac{y}{-\sin y\cos y(1+(\cos y)^2)^{-1/2} - x}$..........................Isolate y' giving you the solution

$y' = -\frac{y}{\frac{\sin y\cos y}{\sqrt{(1+\cos^2 y}} + x}$.......................................Simplify the right side

It appears from the solutions of the learners that 41 learners out of 79 (51.90%) did not convert the root function to an exponential function ($\sqrt{1+\cos^2 y}$ = $(1+\cos^2 y)^{\frac{1}{2}}$) and $\cos^2 y$ to $(\cos y)^2$ (step 1). The second algebra error appeared when learners found y', though 5 learners out of 18 (27.77%) wrongly found y' as $y' = \frac{-\sin y\cos y(1+(\cos y)^2)^{-1/2} - x}{y}$. The third algebra error appeared when learners were unable to perform step 6, though 6 learners out of 13 (46.15%) did not know how to convert the exponential function to the root function, and the properties of roots.

It is also clear that 45 learners out of 79 (56.96%) made the first calculus error in step 2; they differentiated the function in a direct way, though they wrongly differentiated $(1+(\cos y)^2)^{1/2}$ as $(0+(-\sin y)^2)^{1/2}$ or

$\frac{-\sin^2 y'}{2\sqrt{1+\cos^2 y}}$. The second calculus error appeared when learners differentiated implicit functions; 34 learners out of 79 (43.04%) wrote the derivative of $\sqrt{1+\cos^2 y}$ = $(1+\cos^2 y)^{\frac{1}{2}}$ as $(0+(-\sin y)^2)^{1/2} = -\sin y$ or $\frac{1}{2}(-\sin y)^2)^{-1/2} = \frac{1}{2}(-\sin y)^{-1}$, $\frac{-2\sin y'}{2\sqrt{1+\cos^2 y}}$, and the derivative of y as 1, though they did not apply the chain rule or wrongly applied it. The third calculus error appeared when learners differentiated xy, though 32 learners out of 79 (40.51%) did not apply the multiplication rule, so they differentiated it as $(1)(y') = y'$.

DISCUSSION AND RECOMMENDATIONS

Examining and analyzing the learners' answers showed that there are two types of errors that most of the learners made, namely Algebra Errors (AE) and Calculus Errors (CE). Algebra Errors appeared when learners isolated the common factor with average percentages of 34.70, collected similar terms with average percentages of 31.55, simplified Algebra fractions with average percentages of 47.92, multiplied Algebra terms with average percentages of 37.69, dealt with exponentials and roots functions with average percentages of 50.35, and found y' with average percentages of 28.66.

These results are partially consistent with the results of Siyepu (2015), Bicer *et al.* (2014), Muzangwa and Chifamba (2012), Ciltas and Tatar (2011), Alshare' and Alabed (2010), Orhun (2010), Alqudah (2008), and Engelbrecht *et al.* (2005) that indicated that learners commit previous Algebra errors when solving problems in mathematics in general, and in Calculus and implicit differentiation in particular. These errors may be due to teaching methods that focus on procedures, and textbooks that focus on procedural questions (Bezuidenhout, 2001), or due to the learners' neglect.

So, it is necessary to focus on these common Algebra errors when teaching mathematics, Calculus and implicit differentiation, and repeating questions that contain such errors so that learners master these Algebra procedures (Ureyen *et al.*, 2006). Teachers must be well aware of such errors and start with these errors when teaching implicit differentiation or any other topic of Calculus (Bezuidenhout, 2001), as learners learn from their errors (Melis, 2003). This helps them to succeed in their tasks (Dawkins and Epperson, 2014).

The analysis of learners' answers also revealed another type of learners' common errors, which is the Calculus Errors that appeared when learners applied the chain rule with average percentages of 49.33, applied a multiplication rule with average percentages of 37.01, found the derivative of any constant with average percentages of 24.94, differentiated functions with average percentages of 47.49.

These results are partially consistent with Chu (2019), Mirin and Zazkis (2019), Jones (2017), Kakoma and Makonye (2010), Infante (2007) and Martin (2000). These errors may be due to students' lack of understanding of the chain rule and differentiating two multiplied functions (Kakoma and Makonye (2010), Infante (2007) and Martin (2000), or due to learners' inattention and their lack of focus with their teachers, or it might be due to teachers' focus on procedural processes and neglect to explain concepts and the foundations of these rules Chu (2019), Mirin and Zazkis (2019) and Jones (2017).

So, teachers must focus on these Calculus errors when teaching implicit differentiation, solving many questions on one idea, involving all learners in solving processes inside the classrooms, and giving questions as an application to the lesson as homework and paying attention to correcting them and providing immediate feedback to learners (Ureyen *et al.*, 2006). Teachers must also recognize the errors of learners in detail as they mentioned in the findings part of this study, along with paying attention to them when teaching implicit differentiation and emphasizing them in order not to be committed by students in the future.

CONCLUSION

This study aimed to analyze learners' errors in implicit differentiation and identify learners' common errors committed by 20% or more of them. So, the researcher prepared a test in implicit differentiation consisting of seven various questions covering almost all ideas of implicit differentiation in the textbook (Differential Calculus Math101), then the researcher examined the learners' answers to these questions accurately, and identify learners' common errors. By analyzing the learners' answers, the researcher concluded two types of learners' common errors, namely: Algebra Errors (AE) and Calculus Errors (CE). Algebra Errors appeared when learners isolated the common factor, collected similar terms, simplified Algebra fractions, multiplied Algebra terms, dealt with exponentials and roots functions, and found y', and Calculus Errors appeared when learners applied the chain rule, applied a multiplication rule, found the derivative of any constant, and differentiated functions.

So teachers must be well aware of such errors and start with these errors when teaching implicit differentiation or any other topic of Calculus and focus on these Calculus errors when teaching implicit differentiation, solving many questions on one idea, involving all learners in solving processes inside the classrooms, and giving questions as an application to the lesson as homework and paying attention to correcting them and providing immediate feedback. They also must recognize these common errors and paying them attention when teaching implicit differentiation and emphasizing them in order not to be committed by students in the future.

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